COMPUTING PI:
A BRIEF HISTORY

Steven J. Wilson

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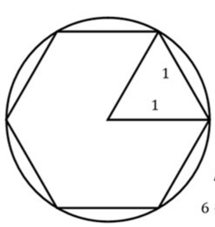
Three Phases

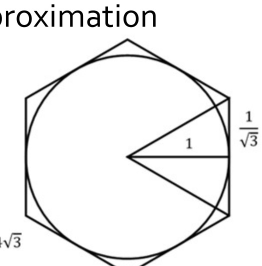
- Geometric Period (until ~1650)
  - Polygon side ratios and geometry: Archimedes
  - Polygon side lengths with algebra or trigonometry
  - Polygon areas and geometric series: Liu Hui
  - Trigonometric bounds: Snell
- Analytic Period (1650-1975)
  - Arctangent power series: Leibniz, Madhava, Sharp
  - Sums of power series: Machin & others
- Modern Approaches (since 1975)
  - Arithmetic-Geometric Mean: Salamin, Brent
  - Ramanujan's Notebook: Borwein, Gosper, Chudnovsky
  - Digit-Extraction Algorithms: Bailey-Borwein-Plouffe

First Geometric Approximation



$r = 1$
$d = 2$

$\pi = \frac{C}{d}$



$P_I < C < P_c$

$6 < C < \frac{12}{\sqrt{3}} = 4\sqrt{3}$

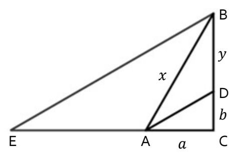
$3 < \pi < 2\sqrt{3} \approx 3.464$

- Archimedes began here, then doubled the number of sides.
- He did NOT use algebra (or trigonometry) or decimals.

Archimedes (~250 BC) used Euclid

- The angle bisector divides the opposite side of a triangle in the same ratio as the sides adjacent to the angle.
- Proof: Given $\triangle ABC$, with AD the bisector of $\angle BAC$.
  - Extend CA to point E so that $AB = AE$.
  - Then $\triangle ABE$ is isosceles, so $\angle ABE = \angle E$.
  - Then alternate exterior angle $\angle BAC = \angle ABE + \angle E = 2\angle E$.
  - And $\angle DAC = \angle E$, so $\triangle ACD$ and $\triangle ECB$ are similar.
  - Then $\frac{AC}{DC} = \frac{EC}{BC} = \frac{AB+A}{BC}$.
  - So $\frac{AB+AC}{AC} = \frac{BD+DC}{DC}$, then $\frac{AB}{AC} = \frac{BD}{DC}$.

In particular: $\frac{a}{b} = \frac{a+x}{b+y}$



Archimedes: Circumscribed Ratios

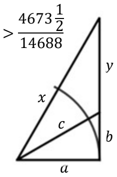
- Let $r_n = \frac{a}{b+y}$ and $s_n = \frac{x}{b+y}$
- Then $r_{2n} = \frac{a}{b} = \frac{a+x}{b+y} = r_n + s_n$ and $s_{2n} = \frac{c}{b} = \sqrt{\frac{b^2+a^2}{b^2}} = \sqrt{1+{r_{2n}}^2}$
- Beginning with a circle of radius 153 …

n	r	s
6	> 265/153	306/153
12	> 571/153	> (591+1/8)/153
24	> (1162+1/8)/153	> (1172+1/8)/153
48	> (2334+1/4)/153	> (2339+1/4)/153
96	> (4673+1/2)/153	

For a 96-gon:

$$\frac{diameter}{perimeter} = \frac{2a}{96(2b)} = \frac{r_{96}}{96} > \frac{4673\frac{1}{2}}{14688}$$

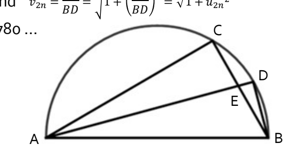
$$\pi = \frac{C}{d} < \frac{14688}{4673\frac{1}{2}} < 3\frac{1}{7}$$



Archimedes: Inscribed Ratios

- Since $\triangle ACE$ and $\triangle ADB$ are similar, we have $\frac{AD}{BD} = \frac{AC}{EC}$.
- Since AE bisects $\angle CAB$, then $\frac{AC}{EC} = \frac{AB+A}{BC}$.
- Let $u_n = \frac{AC}{BC}$ and $v_n = \frac{AB}{BC}$
- Then $u_{2n} = \frac{AD}{BD} = \frac{AC+AB}{BC} = u_n + v_n$ and $v_{2n} = \frac{AB}{BD} = \sqrt{1+\left(\frac{AD}{BD}\right)^2} = \sqrt{1+{u_{2n}}^2}$
- Beginning with a circle of radius 780 …

n	u	v
6	< 1351/780	1560/780
12	< 2911/780	< (3013+3/4)/780
24	< 1823/240	< (1838+9/11)/240
48	< 1007/66	< (1009+1/6)/66
96	< (2016+1/6)/66	< (2017+1/4)/66



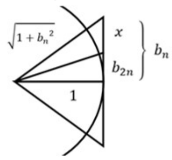
$$\pi > \frac{perimeter}{diameter} = \frac{96}{v_{96}} = \frac{6336}{2017\frac{1}{4}} > 3\frac{10}{71}$$

Using side lengths and algebra

- Circumscribed

$$\frac{x}{b_{2n}} = \frac{\sqrt{1+{b_n}^2}}{1}$$

$$x = b_n - b_{2n}$$



$$b_{2n} = \frac{b_n}{1+\sqrt{1+{b_n}^2}}$$

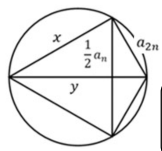
$$Perimeter = 2nb_n$$

- Inscribed

$${a_{2n}}^2 + x^2 = 4$$

$$y^2 + \left(\frac{1}{2}a_n\right)^2 = x^2$$

$$(2-y)^2 + \left(\frac{1}{2}a_n\right)^2 = {a_{2n}}^2$$



$$4 - {a_{2n}}^2 = y^2 + \frac{1}{4}{a_n}^2$$

$$y = 2 - \sqrt{{a_{2n}}^2 - \frac{1}{4}{a_n}^2}$$

$${a_{2n}}^4 - 4{a_{2n}}^2 + {a_n}^2 = 0$$

$$a_{2n} = \sqrt{2-\sqrt{4-{a_n}^2}}$$

$$Perimeter = na_n$$

Using side lengths and algebra

$$a_{2n} = \sqrt{2-\sqrt{4-{a_n}^2}}$$

$$Perimeter = na_n$$

$$\frac{na_n}{2} < \pi < nb_n$$

$$b_{2n} = \frac{b_n}{1+\sqrt{1+{b_n}^2}}$$

$$Perimeter = 2nb_n$$

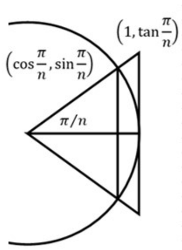
Sides	Inscribed length	Circum length	Inscr Perimeter	Circum perimeter
6	1	0.577350269	3	3.464101615
12	0.51763809	0.267949192	3.105828541	3.215390309
24	0.261052384	0.131652498	3.132628613	3.159659942
48	0.130806258	0.065543463	3.139350203	3.146086215
96	0.065438166	0.03273661	3.141031951	3.1427146
192	0.032723463	0.016363922	3.141452472	3.14187305
384	0.016362279	0.008181413	3.141557608	3.141662747
768	0.008181208	0.004090638	3.141583892	3.141610177
1536	0.004090613	0.002045311	3.141590463	3.141597034
3072	0.002045307	0.001022654	3.141592106	3.141593749
6144	0.001022654	0.000511327	3.141592517	3.141592927
12288	0.000511327	0.000255663	3.141592619	3.141592722
24576	0.000255663	0.000127832	3.141592645	3.141592671

Polygons with Trigonometry

- Central angle $\frac{2\pi}{n}$
- Inscribed $P_I = 2n\sin\frac{\pi}{n}$
- Circumscribed $P_C = 2n\tan\frac{\pi}{n}$

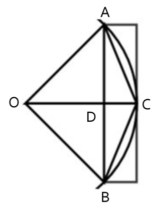
$$P_I < C < P_c$$

$$n\sin\frac{\pi}{n} < \pi < n\tan\frac{\pi}{n}$$



Sides	Inscr Perimeter	Circum Perimeter
3	2.598076211	5.196152423
4	2.828427125	4
5	2.938926261	3.63271264
6	3	3.464101615
12	3.105828541	3.215390309
24	3.132628613	3.159659942
48	3.139350203	3.146086215
56	3.139945045	3.144892543
57	3.14000234	3.144777633
96	3.141031951	3.1427146
159	3.141388247	3.142001539
160	3.141390794	3.141996443
1186	3.14158898	3.141600001
1187	3.141588986	3.141599989
1395	3.141589998	3.141597965
1396	3.141590002	3.141597957
5466	3.141592481	3.141593
5467	3.141592481	3.141592999
15004	3.141592631	3.1415927
15005	3.141592631	3.141592699

Liu Hui (263 AD)



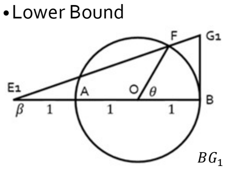
- An n-gon with one side AB of length a_n is inscribed in a unit circle with center O.
- The apothem OD has length $r_n = \sqrt{1 - \frac{1}{4}{a_n}^2}$
- The n-gon has area $A_n = \frac{n}{2} r_n a_n$
- Side AC of a 2n-gon has length $a_{2n} = \sqrt{\frac{{a_n}^2}{4} + (1 - r_n)^2}$
- The difference in areas of the 2n-gon and n-gon is $D_{2n} = A_{2n} - A_n$
- Liu Hui's Inequality: $A_{2n} < \pi < A_{2n} + D_{2n}$
- The ratio $\frac{D_{2n}}{D_n} \to \frac{1}{4}$, hence $D_{2n} + D_{4n} + D_{8n} \ldots \approx \frac{1}{3} D_n$
- Therefore $\pi \approx A_n + \frac{1}{3} D_n$

Liu Hui

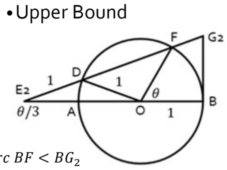
Sides	Insc Length	Polyg radius	Inscr Area	Area Diff	Diff Ratio	Pi Approx
6	1	0.866025404	2.598076211			
12	0.51763809	0.965925826	3	0.401923789		
24	0.261052384	0.991444861	3.105828541	0.105828541	0.263304995	3.141104722
48	0.130806258	0.997858923	3.132628613	0.026800072	0.253240494	3.141561971
96	0.065438166	0.999464587	3.139350203	0.00672159	0.250804914	3.141590733
192	0.032723463	0.999866138	3.141031951	0.001681748	0.250200905	3.141592534
384	0.016362279	0.999966534	3.141452472	0.000420521	0.250050206	3.141592646
768	0.008181208	0.999991633	3.141557608	0.000105136	0.25001255	3.141592653
1536	0.004090613	0.999997908	3.141583892	2.62842E-05	0.250003137	3.141592654
3072	0.002045307	0.999999477	3.141590463	6.57108E-06	0.250000784	3.141592654
6144	0.001022654	0.999999869	3.141592106	1.64277E-06	0.250000196	3.141592654
12288	0.000511327	0.999999967	3.141592517	4.10693E-07	0.250000049	3.141592654
24576	0.000255663	0.999999992	3.141592619	1.02673E-07	0.250000012	3.141592654

Snell's Inequality (1621)

- Lower Bound



- Upper Bound

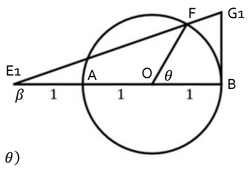


$$BG_1 < arc\ BF < BG_2$$

$$\frac{3 \sin\theta}{2 + \cos\theta} < \theta < 2 \sin\frac{\theta}{3} + \tan\frac{\theta}{3}$$

$$n\left(\frac{3 \sin\frac{\pi}{n}}{2 + \cos\frac{\pi}{n}}\right) < \pi < n\left(2 \sin\frac{\pi}{3n} + \tan\frac{\pi}{3n}\right)$$

Snell's Inequality: Lower Bound



$\tan\beta = \frac{BG_1}{3}$

- Law of Sines on ΔE_1OF

  $\frac{\sin\beta}{1} = \frac{\sin(\pi-\theta)}{E_1F} = \frac{\sin\theta}{E_1F}$
- Law of Cosines on ΔE_1OF

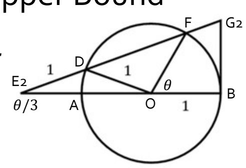
  $E_1F^2 = 2^2 + 1^2 - 2(2)(1)\cos(\pi-\theta)$

  $E_1F = \sqrt{5+4\cos\theta}$
- Therefore $\sin\beta = \frac{\sin\theta}{\sqrt{5+4\cos\theta}}$

  $\cos\beta = \frac{2+\cos\theta}{\sqrt{5+4\cos\theta}}$

  $BG_1 = 3\tan\beta = \frac{3\sin\theta}{2+\cos\theta}$

Snell's Inequality: Upper Bound



- Isosceles triangles ΔDE_2O and ΔODF

  $\angle DE_2O = \frac{\theta}{3}$
- Therefore

  $\tan\frac{\theta}{3} = \frac{BG_2}{BE_2}$

  $BE_2 = 1 + 2\cos\frac{\theta}{3}$

  $BG_2 = \left(1+2\cos\frac{\theta}{3}\right)\tan\frac{\theta}{3} = 2\sin\frac{\theta}{3} + \tan\frac{\theta}{3}$
- A modern approach to finish proving both bounds: Note that $f(0) = 0$, $f'(0) = 1$, and analyze $f(\theta)$ for direction and concavity.

Snell's Inequality: Proof

Lower Bound

- Let $f(\theta) = \frac{3\sin\theta}{2+\cos}$, so $f(0) = 0$.
- $f'(\theta) = \frac{6\cos\theta+3}{(2+\cos\theta)^2}$, so $f'(0) = 1$.
- $f''(\theta) = \frac{-6\sin\ (1-\cos\theta)}{(2+\cos\theta)^3}$.

Then on the interval $\left(0, \frac{\pi}{2}\right)$

- $f'(\theta) > 0$, so $f(\theta)$ is increasing.
- $f''(\theta) < 0$, so $f(\theta)$ is concave down.
- Implying $f(\theta) < \theta$.

- Upper Bound
- Let $f(\theta) = 2\sin\frac{\theta}{3} + \tan\frac{\theta}{3}$, so $f(0) = 0$.
- $f'(\theta) = \frac{2}{3}\cos\frac{\theta}{3} + \frac{1}{3}\sec^2\frac{\theta}{3}$, so $f'(0) = 1$.
- $f''(\theta) = \frac{2}{9}\sin\frac{\theta}{3}(\sec^3\theta - 1)$
- Then on the interval $\left(0, \frac{\pi}{2}\right)$
- $f'(\theta) > 0$, so $f(\theta)$ is increasing.
- $f''(\theta) > 0$, so $f(\theta)$ is concave up.
- Implying $f(\theta) > \theta$.

Therefore: $\frac{3\sin\theta}{2+\cos\theta} < \theta < 2\sin\frac{\theta}{3} + \tan\frac{\theta}{3}$

With Snell

n	Lower Bound	Upper Bound
3	3.117691454	3.144031563
4	3.1344465	3.142349131
5	3.138741703	3.141899717
6	3.140237343	3.141740016
7	3.140867392	3.141671964
8	3.141169899	3.141639056
9	3.141329745	3.141621585
10	3.141420634	3.141611618
11	3.141475401	3.141605598
12	3.141509994	3.141601788
13	3.141532713	3.141599283
28	3.141589883	3.141592961
29	3.141590247	3.141592921
40	3.141591989	3.141592727
41	3.141592052	3.14159272
74	3.141592597	3.14159266
75	3.1415926	3.14159266
76	3.141592603	3.141592659
96	3.141592634	3.141592656

Without Snell

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3	2.598076211	5.196152423
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Early Record Holders

When?	Who?	Decimals?	How?
~250 BC	Archimedes	2	Polygon side ratios, 3×2^5 sides
150 AD	Ptolemy	3	Polygon side ratios
263	Liu Hui	5	Polygon areas, 3×2^5 sides
480	Zu Chongzhi	7	Polygon areas, 3×2^{12} sides
1424	Jamshid al-Kashi	16	Polygon side ratios, 3×2^{28} sides
1596	Ludolph von Ceulen	20	Polygon side ratios, 60×2^{33} sides
1615	Ludolph von Ceulen	32	Polygon side ratios, 2^{62} sides
1621	Willebrord Snell	35	Snell's Inequality, 2^{30} sides
1630	Christoph Grienberger	38	Snell's Inequality

Leibniz Series (1674)

- Geometric Series: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$
- Then $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$
- And $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$
- But $\frac{\pi}{4} = \arctan 1$, therefore: $\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = \sum (-1)^n \frac{4}{2n+1}$
- This converges by the Alternating Series Test, but very slowly!
- Error bound: $|E_n| \leq |a_{n+1}| = \frac{4}{2n+3}$, so need (roughly) $n > \frac{2}{E}$
  - For 2 decimal places (.005 accuracy), need approximately 400 terms
  - For 5 decimal places, need approximately 400000 terms

Madhava (~1400) & Sharp (1699)

- Given
  $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots$
- Using $\frac{\pi}{6} = \arctan \frac{1}{\sqrt{3}}$
- Then $\pi = 6 \sum \frac{(-1)^n}{2n+1} \left(\frac{1}{\sqrt{3}}\right)^{2n+}$
- Or $\pi = \sqrt{12} \sum \frac{(-1)^n}{2n+} \left(\frac{1}{3}\right)^n$
- Error $|E_n| \le \frac{\sqrt{12}}{2n+3} \left(\frac{1}{3}\right)^{n+1}$

Index	Term	Partial Sum	Error	ErrorBound
	0	0		
0	3.464101615	3.464101615	0.322508962	3.85E-01
1	-0.384900179	3.079201436	0.062391218	7.70E-02
2	0.076980036	3.156181472	0.014588818	1.83E-02
3	-0.01832858	3.137852892	0.003739762	4.75E-03
4	0.004751854	3.142604746	0.001012092	1.30E-03
5	-0.00129596	3.141308785	0.000283868	3.66E-04
6	0.000365527	3.141674313	8.16591E-05	1.06E-04
7	-0.000105597	3.141568716	2.39376E-05	3.11E-05
8	3.10579E-05	3.141599774	7.12022E-06	9.26E-06
9	-9.26287E-06	3.141590511	2.14265E-06	2.79E-06
10	2.79357E-06	3.141593305	6.50913E-07	8.50E-07
11	-8.50215E-07	3.141592454	1.99302E-07	2.61E-07
12	2.60733E-07	3.141592715	6.14306E-08	8.05E-08
13	-8.04731E-08	3.141592635	1.90425E-08	2.50E-08
14	2.49744E-08	3.14159266	5.93192E-09	7.79E-09

John Machin (~1686-1751)

- Machin's Formula (1706): $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$

Index	Term	PartialSum	Error	ErrorBound
	0	0		
0	3.183263598	3.183263598	4.17E-02	4.27E-02
1	-0.042666569	3.140597029	9.96E-04	1.02E-03
2	0.001024	3.141621029	2.84E-05	2.93E-05
3	-2.92571E-05	3.141591772	8.81E-07	9.10E-07
4	9.10222E-07	3.141592682	2.88E-08	2.98E-08
5	-2.97891E-08	3.141592653	9.74E-10	1.01E-09
6	1.00825E-09	3.141592654	3.38E-11	3.50E-11

An Arctangent Addition Formula

- Since $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$
- Then $\alpha + \beta = \arctan \left(\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}\right)$
- But also $\alpha + \beta = \arctan \tan \alpha + \arctan \tan \beta$

- Now let $\tan \alpha = \frac{a_1}{b_1}$ and $\tan \beta = \frac{a_2}{b_2}$
- Then $\arctan \frac{a_1}{b_1} + \arctan \frac{a_2}{b_2} = \arctan \frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2}$

Formula of Machin (1706)

- Using $\arctan\frac{a_1}{b_1} + \arctan\frac{a_2}{b_2} = \arctan\frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2}$
- Then $2\arctan\frac{1}{5} = \arctan\frac{1(5)+1(5)}{5(5)-1(1)} = \arctan\frac{10}{24} = \arctan\frac{5}{12}$
- And $4\arctan\frac{1}{5} = \arctan\frac{5(12)+5(12)}{12(12)-5(5)} = \arctan\frac{120}{119}$
- Since $-\frac{\pi}{4} = \arctan\frac{-1}{1}$
- Then $-\frac{\pi}{4} + 4\arctan\frac{1}{5} = \arctan\frac{-1(119)+120(1)}{1(119)-(-1)(120)} = \arctan\frac{1}{239}$
- Therefore $\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$
- As a series: $\pi = \sum \frac{(-1)^n}{2n+1}\left(\frac{16}{5^{2n+1}} - \frac{4}{239^{2n+1}}\right)$
- Error: $|E_n| < \frac{16}{2n+3}\left(\frac{1}{5}\right)^{2n+3}$

Sums of Arctangents

- $\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$ Machin (1706), Shanks (1853)
- $\frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}$ Anon. MS (1721), Vega (1794)
- $\frac{\pi}{4} = 2\arctan\frac{1}{3} + \arctan\frac{1}{7}$ Vega (1789), Clausen (1847)
- $\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{70} + \arctan\frac{1}{99}$ Rutherford (1841)
- $\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}$ Dase (1844)
- $\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$ Lehmann (1853)
- $\frac{\pi}{4} = 3\arctan\frac{1}{4} + \arctan\frac{1}{20} + \arctan\frac{1}{1985}$ Ferguson (1946)
- $\frac{\pi}{4} = 8\arctan\frac{1}{10} - \arctan\frac{1}{239} - 4\arctan\frac{1}{515}$ Felton (1957)
- $\frac{\pi}{4} = 12\arctan\frac{1}{18} + 8\arctan\frac{1}{57} - 5\arctan\frac{1}{239}$ Felton (1958)

More Record Holders

When?	Who?	Decimals?	How?
1400	Madhava	10	Arctangent series for π/6
1699	Abraham Sharp	71	Arctangent series for π/6
1706	John Machin	100	Machin's Formula
1719	Thomas Fantet de Lagny	112	Arctangent series
1721	Anonymous of Philadelphia	152	Arctangent series for π/6, **unpublished**
1794	Jurij Vega	136	Machin-like formula
1841	William Rutherford	152	3-term Machin-like formula
1844	Zacharias Dase	200	3-term Machin-like formula
1853	William Shanks	527	Machin's Formula
1946	D. F. Ferguson	620	3-term Machin-like formula, by hand
1947	D. F. Ferguson	710	3-term Machin-like formula, desk calculator
1949	Levi B. Smith, John Wrench	1120	Machin's Formula, desk calculator
1949	G. W. Reitwiesner et al.	2037	Machin's Formula, ENIAC computer
1957	George E. Felton	7480	3-term Machin-like formula, computer
1961	Daniel Shanks, John Wrench	100265	3-term Machin-like formula, computer
1973	J. Guilloud, M. Bouyer	1 million	3-term Machin-like formula, supercomputer

Rapidly Converging Sequences

- Arithmetic Mean: $AM(a,b) = \frac{a+b}{2}$
- Geometric Mean: $GM(a,b) = \sqrt{ab}$ when a,b are positive.
- Arithmetic-Geometric Mean (AGM), for a, b positive:

$$a_{n+1} = \frac{a_n+b_n}{2} = AM(a_n, b_n), \quad \text{with } a_0 = a$$

$$b_{n+1} = \sqrt{a_n b_n} = GM(a_n, b_n), \quad \text{with } b_0 = b$$

$$AGM(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

- Studied by Gauss (1777-1855), but unpublished in his lifetime.
- The AGM always exists, as the iterations always converge.
- Each iteration roughly doubles the number of correct digits.

AGM Convergence Proofs

Convergence, therefore existence of AGM: Assuming $a > b$,

- $a_n^2 - b_n^2 = \frac{1}{4}(a_{n-1} + b_{n-1})^2 - a_{n-1}b_{n-1} = \frac{1}{4}(a_{n-1} - b_{n-1})^2$
- Thus $4(a_n + b_n)(a_n - b_n) = (a_{n-1} - b_{n-1})^2$
- and $\frac{a_n - b_n}{a_{n-1} - b_{n-1}} = \frac{a_{n-1} - b_{n-1}}{4(a_n + b_n)} = \frac{a_{n-1} - b_{n-1}}{2(a_{n-1} + b_{n-1}) + 4b_n} \le \frac{a_{n-1}}{2a_{n-1}} = \frac{1}{2}$
- Therefore $a_n - b_n \le \frac{1}{2}(a_{n-1} - b_{n-1}) \le \cdots \le \left(\frac{1}{2}\right)^n (a - b)$
- and $\lim_{n\to\infty}(a_n - b_n) = 0.$

Rapid convergence:

- Assuming $a > b$, let $c_n = \sqrt{a_n^2 - b_n^2} = \frac{a_{n-1} - b_{n-1}}{2}$
- Thus $4a_n c_n = 4\left(\frac{a_{n-1} + b_{n-1}}{2}\right)\left(\frac{a_{n-1} - b_{n-1}}{2}\right) = a_{n-1}^2 - b_{n-1}^2 = c_{n-1}^2$
- But $a_n = \frac{a_{n-1} + b_{n-1}}{2} < a_{n-1}$, so $AGM(a,b) < a_n$
- Therefore $c_n = \frac{c_{n-1}^2}{4a_n} < \frac{c_{n-1}^2}{4AGM(a,b)} \to 0$ quadratically.

Salamin-Brent Algorithm (1976)

- Let $a_0 = 1$ and $b_0 = \frac{1}{\sqrt{2}}$.
- Define $a_{n+1} = \frac{a_n + b_n}{2}$, $b_{n+1} = \sqrt{a_n b_n}$. (Arithmetic-Geometric Mean)
- Define $c_n = a_n^2 - b_n^2$. (Not Pythagorean)
- Define $s_{n+1} = s_n - 2^{n+1} c_{n+1}$, with $s_0 = \frac{1}{2}$.
- Then $p_n = \frac{2a_n^2}{s_n}$ converges to π. (The proof involves elliptic integrals.)
- Each iteration doubles the number of correct digits.

Index	a	b	c	s	p: Value	Error
0	1	0.707106781	0.5	0.5	4	0.858407346
1	0.853553391	0.840896415	0.021446609	0.457106781	3.187672643	0.046079989
2	0.847224903	0.847201267	4.00498E-05	0.456946582	3.141680293	8.76397E-05
3	0.847213085	0.847213085	1.39667E-10	0.456946581	3.141592654	3.05667E-10
4	0.847213085	0.847213085	0	0.456946581	3.141592654	1.37668E-14

Borwein's Quartic (1983, pub. 1987)

- Let $y_0 = \sqrt{2} - 1$ and $a_0 = 6 - 4\sqrt{2}$.
- Define $y_{n+1} = \frac{1-\sqrt[4]{1-y_n^4}}{1+\sqrt[4]{1-y_n^4}}$.
- Also $a_{n+1} = a_n(1 + y_{n+1})^4 - 2^{2n+3} y_{n+1}(1 + y_{n+1} + y_{n+1}^2)$.
- Then $\frac{1}{a_n}$ converges to π.
- Each iteration quadruples the number of correct digits.
- Bailey used 12 iterations in 1986 to produce 29 million digits of π.
- 25 iterations would produce 1 quadrillion correct digits.

Index	y	a	Reciprocal	Error
0	0.414213562	0.343145751	2.914213562	0.227379091
1	0.003734885	0.318309887	3.141592646	7.37625E-09
2	2.43231E-11	0.318309886	3.141592654	1.33227E-15
3	0	0.318309886	3.141592654	1.33227E-15

Formulas of Ramanujan (1914)

- $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum \frac{(4n)!(1103+26390\ \ )}{(n!)^4 396^{4n}}$
- Each term of this series produces 8 additional digits of π.
- Used by Gosper (1985) to obtain 17 million digits of π, which was briefly a world record.

- $\frac{1}{\pi} = 12 \sum \frac{(-1)^n (6n)!(13591409+545140134\ \ )}{(3n)!(n!)^3 640320^{3n+3/2}}$
- Each term of this series produces 14 additional digits of π.
- Tweaked by the Chudnovsky brothers (1988) for high performance, then used by them to produce 4 billion digits of π in 1989-1994.
- Used by others 2009-2023 to produce trillions of digits of π.

BBP: A Digit-Extraction Algorithm

Bailey, Borwein, and Plouffe (discovered 1995, pub. 1997)

- $\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right)$
  - This can be used to produce hexadecimal digits without knowing all of the previous digits. Suppose we want 6th digit, then multiply by $16^{6-1} = 16^5$.
- Let $S_j = \sum_{i=0}^{\infty} \frac{16^{-i}}{8i+j}$, so $16^5\pi = 4(16^5 S_1) - 2(16^5 S_4) - 16^5 S_5 - 16^5 S_6$.
- Let $a_j = Frac(16^5 S_j) = Frac\left(Frac\left(\sum_{i=0}^{5} \frac{16^{5-i} \bmod (8i+j)}{8i+j}\right) + \sum_{i=6}^{\infty} \frac{16^{5-i}}{8i+j}\right)$
  - Multiplying by 16^5 moves hex point so desired digits are immediately to right. Frac (above floor) & mod eliminate unwanted digits to left for better precision.
  - Note: Our example below computes in base 10 and is converted to hex. In actual use, entire computation would be done in base 16.
- $a_1 = Frac\left(Frac\left(0 + \frac{7}{9} + \frac{16}{17} + \frac{6}{25} + \frac{16}{33} + \frac{1}{41}\right) + \frac{1}{16(49)} + \frac{1}{16^2(57)} + \cdots\right)$
- $a_j = \{0.4695409829, 0.7446330113, 0.7618761585, 0.9686911814\}$
- $Frac(16^5\pi) = Frac(4a_1 - 2a_4 - a_5 - a_6) = 0.6583305691$
- Convert to Hex: $(0.6583305691)_{10} = (0.A8885A283DD76917794)_{16}$
- Compare: $(3.1415926535)_{10} = (3.243F6A8822E87C199ACB)_{16}$

Modern Record Holders

When?	Who?	Decimals?	How?
1985	Bill Gosper	17 million	a Ramanujan formula
1986	David Bailey	29 million	Borwein quartic, Cray supercomputer
1988	Yasumasa Kanada et al.	201 million	Salamin-Brent & Borwein quartic, Fortran
1989	Chudnovsky brothers	1 billion	Chudnovsky algorithm, mainframe computer
1994	Chudnovsky brothers	4 billion	Chudnovsky algorithm, homemade computer
1999	Yasumasa Kanada et al.	200 billion	Salamin-Brent algorithm
2002	Yasumasa Kanada et al.	1.24 trillion	Chudnovsky algorithm, supercomputer
2009	Fabrice Bellard	2.7 trillion	BBP-like algorithm, desktop computer
2010	Shigeru Kondo	5 trillion	Chudnovsky algorithm, y-cruncher, desktop
2013	Shigeru Kondo	12 trillion	Chudnovsky algorithm, y-cruncher
2019	Emma Haruka Iwao	31 trillion	Chudnovsky algorithm, y-cruncher, Google Cloud
2022	Emma Haruka Iwao	100 trillion	Chudnovsky algorithm, y-cruncher, Google Cloud
2024	Jordan Ranous et al.	105 trillion	Chudnovsky algorithm, y-cruncher

A Few Really Good References

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