CONSTRUCTING A LOGARITHM

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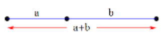
COMPASS AND STRAIGHTEDGE CONSTRUCTIONS

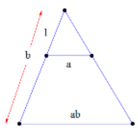
We can:

Add and subtract



- So every positive integer is constructible.

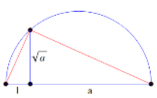
Multiply and divide



- So every positive rational is constructible.

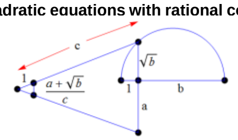
Take a square root

- So every positive square root of a positive rational is constructible.

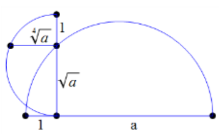


SOME OTHER CONSTRUCTIBLE NUMBERS

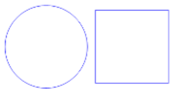
All solutions of quadratic equations with rational coefficients



All fourth roots of rational numbers

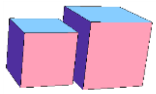


CLASSIC IMPOSSIBLE CONSTRUCTIONS



Squaring the circle

- This would require $\sqrt{\pi}$.

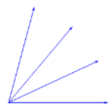


Doubling the cube

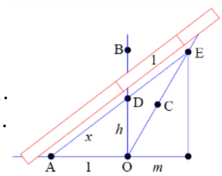
- This would require $\sqrt[3]{2}$.

Trisecting an angle

- Since $\cos(3A) = 4\cos^3 A - 3\cos A$, we would need to solve $4x^3 - 3x = \text{constant}$.



MARKED RULERS



Given: $AO \perp BO, \quad \angle BOC = 30^\circ, \quad AO = 1$**.**

Use marked ruler from A so that $DE = 1$**.**

Then $x = \sqrt[3]{2}$**.**

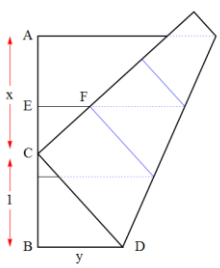
To prove:

By similarity: $\frac{x}{1} = \frac{x+1}{m+1}$ **, so** $m = \frac{1}{x}$ **. Also** $\frac{h}{1} = \frac{\sqrt{3}\,m}{m+1} = \frac{\sqrt{3}}{x+1}$ **.**

By Pythagorean Theorem: $x^2 = h^2 + 1 = \frac{3}{(x+1)^2} + 1$

Solving gives $x = \sqrt[3]{2}$ **.**

ORIGAMI CONSTRUCTIONS



When folded as shown, $x = \sqrt[3]{2}$

To prove:

$$1 + x = BD + CD = y + \sqrt{1 + y^2}$$

$$AE = CF = \frac{x+1}{3}, \quad CE = x - AE = \frac{2x-1}{3}$$

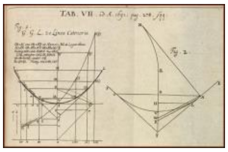
$$\frac{x+1}{2x-1} = \frac{CF}{CE} = \frac{CD}{BD} = \frac{x+1-y}{y}$$

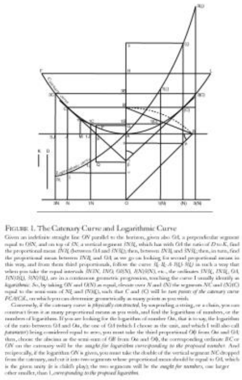
2 equations in 2 variables, solve each for y and substitute:

$$\frac{x^2 + 2x}{2x+2} = y = \frac{2x^2 + x - 1}{3x}$$

$$x = \sqrt[3]{2}$$

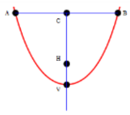
LEIBNIZ, 1691





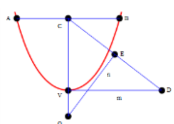
"Conversely, if the catenary curve is physically constructed, by suspending a string, or a chain, you can … find the logarithms of numbers, or the numbers of logarithms …

THE STEPS: HANG A CHAIN



1. **Select point C.**
2. **Let a chain hang vertically from C, and locate point H on the chain below C.**
3. **Construct the perpendicular to CH through point C.**
4. **Locate points A and B on the perpendicular to CH, such that AC = BC, and C is between A and B.**
5. **Hang the chain from points A and B.**
6. **Let V be the intersection of the chain and line CH.**

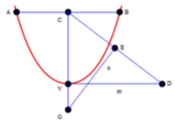
THE STEPS: LOCATE THE ORIGIN

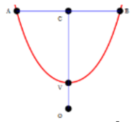


1. **Construct line m perpendicular to CV at V.**
2. **Fix chain at V, remove chain from B and stretch along m to point D, so that DV = arc(BV).**
3. **Let E be the midpoint of CD.**
4. **Let line n be the perpendicular bisector of CD.**
5. **Let O be the intersection of CV and line n.**

INTERLUDE: TAKING STOCK

Points D, E, and lines CD, DV, and EO can now be ignored.



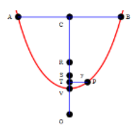


$$y = \cosh x = \frac{e^{x} + e^{-x}}{2}$$

Point O is the origin, and OV = 1.

THE STEPS: FINDING LN 2

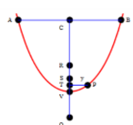
1. **Locate point R on CV so that VR=OV.**
2. **Let S be the midpoint of VR.**
3. **Let T be the midpoint of VS.**
4. **Let line y be the perpendicular bisector of VS.**
5. **Let P be the intersection of the chain and y.**
6. **Then TP has length ln 2.**



CHECKING THE RESULT

If OV = 1, then this implies

$P = (\ln 2, 1.25)$ **and** $\cosh \ln 2 = 1.25$

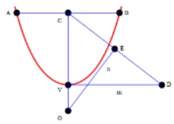


Is this really true?

Recall: $$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

Then: $$\cosh \ln 2 = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4} = 1.25$$

WHY DOES THIS WORK?



When we begin, we know:

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$$

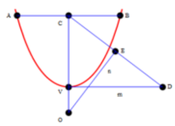
the location of the y-axis

$$f(0) = \cosh 0 = \frac{e^0 + e^{-0}}{2} = 1 \;, \quad \text{so} \quad V = (0,1)$$

But we don't know:

the location of the x-axis (and the origin)

ARC LENGTH



Recall: $\text{Arc Length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2} \qquad f'(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

If point B has coordinates $(x, \cosh x)$ **:**

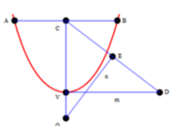
$$\text{Arc Length} = \int_0^x \sqrt{1 + \sinh^2 t}\; dt = \int_0^x \cosh t\; dt = \sinh t\Big|_0^x = \sinh x$$

$$\text{Arc Length} = \int_0^x \sqrt{1 + \left(\frac{e^t - e^{-t}}{2}\right)^2}\; dt = \int_0^x \left(\frac{e^t + e^{-t}}{2}\right) dt = \left(\frac{e^t - e^{-t}}{2}\right)\Bigg|_0^x = \frac{e^x - e^{-x}}{2}$$

So DV has length: $\sinh x = \frac{e^x - e^{-x}}{2}$

COORDINATIZING

Given $BC = x$ **, we have:**



$$A = (-x, \cosh x)$$
$$B = (x, \cosh x)$$
$$C = (0, \cosh x)$$
$$V = (0,1)$$
$$D = (\sinh x, 1)$$

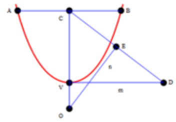
Subtracting coordinates: $CV = \cosh x - 1$

By Pythagorean Theorem: $CD^2 = \sinh^2 x + (\cosh x - 1)^2$

SIMILAR TRIANGLES

But right triangles CVD and CEO are similar.

$$\frac{CO}{CE} = \frac{CD}{CV}$$



And $CE = \frac{1}{2} CD$

Therefore

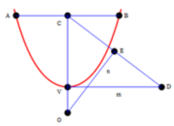
$$CO = \frac{CD^2}{2CV} = \frac{\sinh^2 x + (\cosh x - 1)^2}{2(\cosh x - 1)} = \frac{2\cosh^2 x - 2\cosh x}{2(\cosh x - 1)} = \cosh x$$

And this implies that point O is the origin.

ALTERNATIVELY

Using exponentials:

$$CO = \frac{CD^2}{2CV} = \frac{\left(\frac{e^x - e^{-x}}{2}\right)^2 + \left(\frac{e^x + e^{-x}}{2} - 1\right)^2}{2\left(\frac{e^x + e^{-x}}{2} - 1\right)}$$



$$= \frac{\left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right) + \left(\frac{e^{2x} + 2 + e^{-2x} - 4e^x - 4e^{-x} + 4}{4}\right)}{2\left(\frac{e^x + e^{-x}}{2} - 1\right)}$$

$$= \frac{2\left(\frac{e^{2x} + 1 - 2e^x + 1 + e^{-2x} - 2e^{-x}}{4}\right)}{2\left(\frac{e^x + e^{-x}}{2} - 1\right)}$$

$$= \frac{2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - 1\right)}{2\left(\frac{e^x + e^{-x}}{2} - 1\right)} = \frac{e^x + e^{-x}}{2}$$

GENERALIZING

Given any positive rational number q.

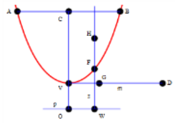
Then: $$\cosh \ln q = \frac{e^{\ln q} + e^{-\ln q}}{2} = \frac{q + \frac{1}{q}}{2} = \frac{q}{2} + \frac{1}{2q}$$

is also rational, and thus constructible.

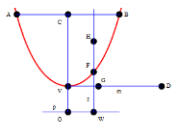
Therefore: The natural logarithm of every positive rational number is constructible (with a chain).

BONUS: FINDING E

1. Construct line p perpendicular to OV through point O.
2. Locate point W on line x so that OV=OW.
3. Construct line z perpendicular to OW through point O.
4. Let F be the intersection of line z and the chain.
5. Fix chain at V, remove chain from B and stretch along m to point G, so that GV = arc(FV).
6. Locate point H on line z so that FH=GV and F is between W and H.
7. Then HW has length e.



VERIFYING THE RESULT



$$FW = \cosh 1 = \frac{e^1 + e^{-1}}{2}$$

$$FH = GV = arc(FV) = \sinh 1 = \frac{e^1 - e^{-1}}{2}$$

$$HW = \sinh 1 + \cosh 1 = e^1$$

THE CATENARY…

"… the catenary,
this marvelous graceful thing,
this joy of physics,
this perfect balance between rebellion and obedience,
is God's own signature on earth."

- Reverend Mootfowl,
in Mark Helprin's *Winter's Tale*, 1983

Presentation at:
http://www.milefoot.com/about/presentations/ConstructingLogs.htm