

# CONSTRUCTING A LOGARITHM

STEVEN J. WILSON  
SWILSON@JCCC.EDU

KAMATYC, WICHITA, KANSAS, MARCH 2015

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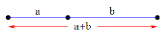
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## COMPASS AND STRAIGHTEDGE CONSTRUCTIONS

We can:

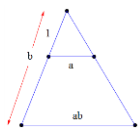
### Add and subtract

- So every positive integer is constructible.



### Multiply and divide

- So every positive rational is constructible.



### Take a square root

- So every positive square root of a positive rational is constructible.




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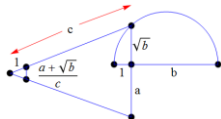
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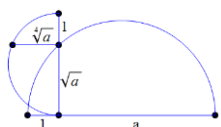
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## SOME OTHER CONSTRUCTIBLE NUMBERS

All solutions of quadratic equations with rational coefficients



All fourth roots of rational numbers




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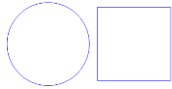
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## CLASSIC IMPOSSIBLE CONSTRUCTIONS

### Squaring the circle

- This would require  $\sqrt{\pi}$ .



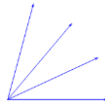
### Doubling the cube

- This would require  $\sqrt[3]{2}$ .



### Trisecting an angle

- Since  $\cos(3A) = 4\cos^3 A - 3\cos A$ , we would need to solve  $4x^3 - 3x = \text{constant}$ .




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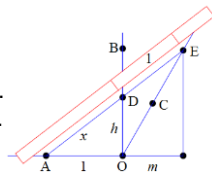
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## MARKED RULERS

Given:  $AO \perp BO$ ,  $\angle BOC = 30^\circ$ ,  $AO = 1$ .  
 Use marked ruler from A so that  $DE = 1$ .  
 Then  $x = \sqrt[3]{2}$ .



To prove:

By similarity:  $\frac{x}{1} = \frac{x+1}{m+1}$ , so  $m = \frac{1}{x}$ . Also  $\frac{h}{1} = \frac{\sqrt{3}m}{m+1} = \frac{\sqrt{3}}{x+1}$ .

By Pythagorean Theorem:  $x^2 = h^2 + 1 = \frac{3}{(x+1)^2} + 1$

Solving gives  $x = \sqrt[3]{2}$ .

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## ORIGAMI CONSTRUCTIONS

When folded as shown,  $x = \sqrt[3]{2}$

To prove:

$$1+x = BD + CD = y + \sqrt{1+y^2}$$

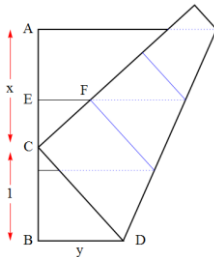
$$AE = CF = \frac{x+1}{3}, \quad CE = x - AE = \frac{2x-1}{3}$$

$$\frac{x+1}{2x-1} = \frac{CF}{CE} = \frac{CD}{BD} = \frac{x+1-y}{y}$$

2 equations in 2 variables, solve each for y and substitute:

$$\frac{x^2 + 2x}{2x+2} = y = \frac{2x^2 + x - 1}{3x}$$

$$x = \sqrt[3]{2}$$




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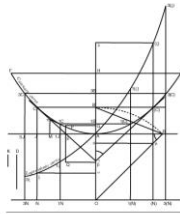
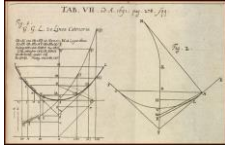
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# LEIBNIZ, 1691



**“Conversely, if the catenary curve is physically constructed, by suspending a string, or a chain, you can ... find the logarithms of numbers, or the numbers of logarithms ...**

**FIGURE 1. The Catenary Curve and Logarithmic Curve**  
 Given an arbitrary straight line  $OP$  parallel to the horizon, given also  $OC$  a perpendicular segment laid on  $OP$ , and an arc of  $OC$  as radius against  $OC$ , which has with  $OC$  the center of  $D$ ,  $OC$ , find the perpendicular mean  $CH$  between  $OC$  and  $OP$ ; draw  $CH$  to the center  $D$ , and  $CH$  to the center  $D$ , and the perpendicular mean between  $OC$  and  $CH$  to the center  $E$ , and so on, by taking the normal proportional mean to each one, and from these third perpendiculars, which are  $OC, CH, EH, IG, KL, MN, OP$ , to each one, lay out their ordinates to the equal arches  $BC, CD, DE, EF, FG, GH, HI, IL, LM, LN, NO$ , in the ordinates  $OC, CH, EH, IG, KL, MN, OP$ , and in consequence geometric progression, knowing the curve I usually denote by logarithmic. On laying  $OC$  and  $CH$  as equal divisions of  $OC$  and  $OP$  against  $OC$  and  $CH$  laid to the same rate of  $OC$  and  $CH$ , such that  $C$  and  $H$  will be just parts of the catenary curve  $BCD$ , which hyperbolic becomes geometrically as every point is proved.  
 Conversely, if the catenary curve is physically constructed by suspending a string or a chain, you can compare  $OC$  to the same proportional mean as you wish, and find the logarithm of numbers, or the number of logarithms. If you are willing to the algorithm, consider the case to be the logarithm of the ratio between  $OC$  and  $CH$ , the rate of  $OC$  which I choose to be the unit, and which I will call  $OC$  (although being a constant equal to one, you must take the third proportional  $CH$  from  $OC$  and  $CH$ ); thus, choose the distance of the same rate of  $CH$  from  $OC$  and  $OP$ , the corresponding ordinate  $BC$  or  $OP$ , and the contrary will be the length of the logarithm corresponding to the proposed number. And reciprocally, if the logarithm  $OC$  is given, you may take the height of the normal segment  $CH$  to equal from the given unity, and so on, in consequence, whose proportional mean should be equal to  $OC$ , which is the given unity to which  $OC$  is the same, and which will be the length of the number, or the length of the number, that is, logarithmically, the proposed logarithm.

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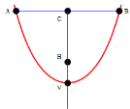
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## THE STEPS: HANG A CHAIN



1. Select point C.
2. Let a chain hang vertically from C, and locate point H on the chain below C.
3. Construct the perpendicular to CH through point C.
4. Locate points A and B on the perpendicular to CH, such that AC = BC, and C is between A and B.
5. Hang the chain from points A and B.
6. Let V be the intersection of the chain and line CH.

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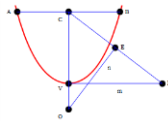
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## THE STEPS: LOCATE THE ORIGIN



1. Construct line m perpendicular to CV at V.
2. Fix chain at V, remove chain from B and stretch along m to point D, so that DV = arc(BV).
3. Let E be the midpoint of CD.
4. Let line n be the perpendicular bisector of CD.
5. Let O be the intersection of CV and line n.

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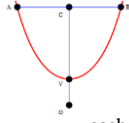
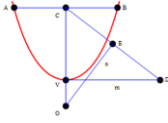
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## INTERLUDE: TAKING STOCK

Points D, E, and lines CD, DV, and EO can now be ignored.



$$y = \cosh x = \frac{e^x + e^{-x}}{2}$$

Point O is the origin, and  $OV = 1$ .

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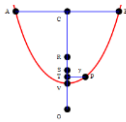
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## THE STEPS: FINDING $\ln 2$

1. Locate point R on CV so that  $VR=OV$ .
2. Let S be the midpoint of VR.
3. Let T be the midpoint of VS.
4. Let line y be the perpendicular bisector of VS.
5. Let P be the intersection of the chain and y.
6. Then TP has length  $\ln 2$ .




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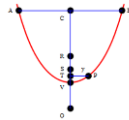
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## CHECKING THE RESULT

If  $OV = 1$ , then this implies

$$P = (\ln 2, 1.25) \text{ and } \cosh \ln 2 = 1.25$$

Is this really true?



Recall:  $\cosh x = \frac{e^x + e^{-x}}{2}$

Then:  $\cosh \ln 2 = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4} = 1.25$

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## WHY DOES THIS WORK?

When we begin, we know:

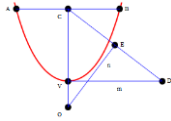
$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$$

the location of the y-axis

$$f(0) = \cosh 0 = \frac{e^0 + e^{-0}}{2} = 1, \text{ so } V = (0,1)$$

But we don't know:

the location of the x-axis (and the origin)




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## ARC LENGTH

Recall: Arc Length =  $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

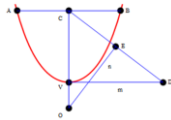
$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2} \quad f'(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

If point B has coordinates  $(x, \cosh x)$ :

$$\text{Arc Length} = \int_0^x \sqrt{1 + \sinh^2 t} dt = \int_0^x \cosh t dt = \sinh t \Big|_0^x = \sinh x$$

$$\text{Arc Length} = \int_0^x \sqrt{1 + \left(\frac{e^t - e^{-t}}{2}\right)^2} dt = \int_0^x \left(\frac{e^t + e^{-t}}{2}\right) dt = \left(\frac{e^t - e^{-t}}{2}\right) \Big|_0^x = \frac{e^x - e^{-x}}{2}$$

So DV has length:  $\sinh x = \frac{e^x - e^{-x}}{2}$




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## COORDINATIZING

Given  $BC = x$ , we have:

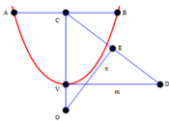
$$A = (-x, \cosh x)$$

$$B = (x, \cosh x)$$

$$C = (0, \cosh x)$$

$$V = (0,1)$$

$$D = (\sinh x, 1)$$



Subtracting coordinates:  $CV = \cosh x - 1$

By Pythagorean Theorem:  $CD^2 = \sinh^2 x + (\cosh x - 1)^2$

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## SIMILAR TRIANGLES

But right triangles CVD and CEO are similar.

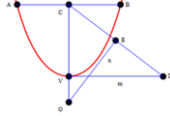
$$\frac{CO}{CE} = \frac{CD}{CV}$$

And  $CE = \frac{1}{2}CD$

Therefore

$$CO = \frac{CD^2}{2CV} = \frac{\sinh^2 x + (\cosh x - 1)^2}{2(\cosh x - 1)} = \frac{2 \cosh^2 x - 2 \cosh x}{2(\cosh x - 1)} = \cosh x$$

And this implies that point O is the origin.




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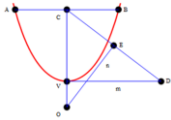
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## ALTERNATIVELY

Using exponentials:

$$\begin{aligned} CO = \frac{CD^2}{2CV} &= \frac{\left(\frac{e^x - e^{-x}}{2}\right)^2 + \left(\frac{e^x + e^{-x}}{2} - 1\right)^2}{2\left(\frac{e^x + e^{-x}}{2} - 1\right)} \\ &= \frac{\left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right) + \left(\frac{e^{2x} + 2 + e^{-2x} - 4e^x - 4e^{-x} + 4}{4}\right)}{2\left(\frac{e^x + e^{-x}}{2} - 1\right)} \\ &= \frac{2\left(\frac{e^{2x} + 1 - 2e^x + 1 + e^{-2x} - 2e^{-x}}{4}\right)}{2\left(\frac{e^x + e^{-x}}{2} - 1\right)} \\ &= \frac{2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - 1\right)}{2\left(\frac{e^x + e^{-x}}{2} - 1\right)} = \frac{e^x + e^{-x}}{2} \end{aligned}$$




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## GENERALIZING

Given any positive rational number q.

Then:  $\cosh \ln q = \frac{e^{\ln q} + e^{-\ln q}}{2} = \frac{q + \frac{1}{q}}{2} = \frac{q}{2} + \frac{1}{2q}$

is also rational, and thus constructible.

Therefore: The natural logarithm of every positive rational number is constructible (with a chain).

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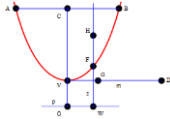
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### BONUS: FINDING E

1. Construct line p perpendicular to OV through point O.
2. Locate point W on line x so that OV=OW.
3. Construct line z perpendicular to OW through point O.
4. Let F be the intersection of line z and the chain.
5. Fix chain at V, remove chain from B and stretch along m to point G, so that  $GV = \text{arc}(FV)$ .
6. Locate point H on line z so that  $FH=GV$  and F is between W and H.
7. Then HW has length e.




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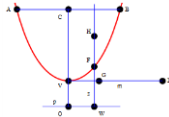
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### VERIFYING THE RESULT



$$FW = \cosh 1 = \frac{e^1 + e^{-1}}{2}$$

$$FH = GV = \text{arc}(FV) = \sinh 1 = \frac{e^1 - e^{-1}}{2}$$

$$HW = \sinh 1 + \cosh 1 = e^1$$

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### THE CATENARY...

“... the catenary,  
this marvelous graceful thing,  
this joy of physics,  
this perfect balance between rebellion and obedience,  
is God's own signature on earth.”



- Reverend Mootfowl,  
in Mark Helprin's *Winter's Tale*, 1983



Presentation at:  
<http://www.milefoot.com/about/presentations/ConstructingLogs.htm>

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