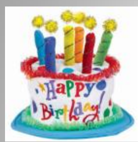

EXTRATERRESTRIAL BIRTHDAYS

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THE BIRTHDAY PROBLEM

- How many people must be in a room before the probability of at least two sharing a birthday is greater than 50%?

SOLUTION

- P(at least 2 share) = 1 – P(no one shares)
- To find P(no one shares), do probabilities of choosing a non-matching birthday:

$$=\frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}\times\cdots\times\left(1-\frac{n-1}{365}\right)$$

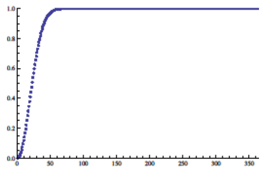
$$=\frac{365\times364\times363\times\cdots\times(365-(n-1))}{365^n}$$

$$=\frac{365!}{(365-n)!\times365^n}=\frac{{}_{365}P_n}{365^n}$$

- So P(at least 2 share) $=1-\frac{{}_{365}P_n}{365^n}$

STANDARD RESULT

- P(at least 2 share) $= 1 - \frac{{}_{365}P_n}{365^n}$



n	P(sharing)
5	2.71%
10	11.69%
15	25.29%
20	41.14%
25	56.87%
30	70.63%
35	81.44%
40	89.12%
45	94.10%
50	97.04%

- For n = 23, P(at least 2 share) = 50.73%

A SERIES APPROXIMATION

- P(no one shares) $= \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{365-(n-1)}{365}$
$$= \left(1 - \frac{0}{365}\right) \times \left(1 - \frac{1}{365}\right) \times \left(1 - \frac{2}{365}\right) \times \cdots \times \left(1 - \frac{n-1}{365}\right)$$

- Recall the Taylor Series: $e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots$

- So a first-order approximation is:
P(no one shares) $= e^{-0/365} \times e^{-1/365} \times e^{-2/365} \times \cdots \times e^{-(n-1)/365}$
$$= 1 \times e^{-(1+2+\cdots+(n-1))/365}$$
$$= e^{-(n(n-1)/2)/365}$$

AN APPROXIMATE SOLUTION

Solving P(at least 2 share) > 0.50

$$1 - e^{-(n(n-1)/2)/365} > 0.5$$
$$e^{-(n(n-1)/2)/365} < 0.5$$
$$\frac{-n(n-1)}{2(365)} < \ln(0.5)$$
$$\boxed{n(n-1) > (365)\ln 4} \approx 505.997$$
$$n^2 - n - 506 > 0$$
$$n > \frac{1 + \sqrt{1 + 4(506)}}{2} = \frac{1 + \sqrt{2025}}{2} = 23$$

OR MORE GENERALLY…

$$n(n-1) > d \ln 4$$

$$n^2 - n - d \ln 4 > 0$$

$$n > \frac{1+\sqrt{1+4d\ln 4}}{2}$$

$$n > 0.5 + \sqrt{d \ln 4}$$

$$n \approx 0.5 + 1.177\sqrt{d}$$

- So n is roughly proportional to the square root of d, with proportionality constant about 1.2.

$$n \approx 1.2\sqrt{365} \approx 22.93$$

$$n \approx 0.5 + 1.177\sqrt{365} \approx 22.99$$

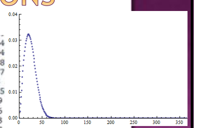
(Pat'sBlog attributes the factor 1.2 to Persi Diaconis)

PROBABILITY DISTRIBUTIONS

The probability of obtaining the first match is highest when n = 20

MODE

	PDF	CDF
10	0.02232434382191 0956	0.116948177711077 64
11	0.0241932006106554 06	0.141141378321733 04
12	0.025883410516331 33	0.167024788838064 38
13	0.0273854863943650 06	0.194410275232429 37
14	0.028692236772543 61	0.223102512004973
15	0.0297988077587133 63	0.252901319763686 35
16	0.030702685489163 57	0.283604005252849 9
17	0.031403660043710 69	0.315007665296560 6
18	0.0319037525752286 85	0.346911417871789 3
19	0.032207108159747 37	0.379118526031536 7
20	0.03231985754904 33	0.41143838358058
21	0.032249951584625 75	0.443688335165205 7
22	0.032006972497344 33	0.475695307662550 1
23	0.031601926661435 34	0.507297234323985 4
24	0.0310470235905433 85	0.538344257914528 8
25	0.0303554460549350 92	0.568699703969463 9
26	0.0295411161664750 76	0.598240820135938 9
27	0.0286184621273029 78	0.626859282263242
28	0.0276021900791574 43	0.654461472342399 4
29	0.026507065135377 58	0.680968537477777
30	0.0253477052414916 93	0.706316242719268 6
31	0.0241383910093751 76	0.730454633728643 8
32	0.0228928941216768 25	0.753347527850320 7
33	0.021624326325451 34	0.774971854175772
34	0.020345010444382 26	0.795316864620154 3
35	0.019066374254560 97	0.814383238874715 2
36	0.0177988675051642 92	0.832182106379879 4
37	0.0165519018365050 33	0.848734008216384 5
38	0.015333812865736 36	0.864067821082120 8
39	0.014151843284601 11	0.878219664366721 8
40	0.0130121454512269 61	0.891231809817948 8



MEDIAN

When n=23, the probability of at least one match first exceeds 50%

MEAN

E(X) ≈ 24.6166

THE FIRST MATCH

- How many trials to obtain the first match?
- Variables:
  - x = number of trials to obtain the first match
  - d = number of equally likely options (birthdays)
- Pattern: Not Not Not … Not Match
- Probability: $\frac{d}{d} \times \frac{d-1}{d} \times \frac{d-2}{d} \times \cdots \times \frac{d-(x-2)}{d} \times \frac{x-1}{d}$

$$P(X=x) = \frac{(x-1)}{d^x}\left({}_dP_{x-1}\right) \quad \text{for} \quad x \in \{2,3,4,\ldots,d+1\}$$

THE MODE ANALYTICALLY

- We avoid calculus, since ${}_dP_{x-1}$ is discrete.
- For which x is P(x) > P(x+1) ?

$$\frac{x-1}{d^x}\left({}_dP_{x-1}\right) > \frac{x}{d^{x+1}}\left({}_dP_x\right)$$
$$\frac{(x-1)}{d^x}\frac{d!}{(d-x+1)!} > \frac{x}{d^{x+1}}\frac{d!}{(d-x)!}$$
$$d(x-1) > x(d-x+1)$$
$$x^2 - x - d > 0$$
$$x > \frac{1+\sqrt{1+4d}}{2}$$

EXACT FORMULA!

- For d = 365, we get x > 19.61, so 19 or 20.
- Approximating: $x > \sqrt{d}$ for large *d*

THE MEAN (EXPECTED VALUE)

- Since $P(X=x) = \frac{(x-1)}{d^x}\left({}_dP_{x-1}\right)$ for $x \in \{2,3,4,\ldots,d+1\}$

  then $E(X) = \sum_{x=2}^{d+1} \frac{x(x-1)}{d^x}\left({}_dP_{x-1}\right)$
- and this is related to Ramanujan's Q-function, specifically $E(X) = 1 + Q(d)$

- which is asymptotic:

$$E(X) \sim 1 + \sqrt{\frac{\pi d}{2}} - \frac{1}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2d}} - \frac{4}{135d} + \cdots$$

- and for d=365, we get:

$$E(X) \approx \sqrt{\frac{365\pi}{2}} + \frac{2}{3} + \frac{1}{12}\sqrt{\frac{\pi}{730}} - \frac{4}{49275} + \cdots \approx 24.6166$$

BIRTHDAY PROBLEM ON JUPITER?

- A Jovian day = 9.9259 Earth-hours
- A Jovian year = 11.86231 Earth-years

$$1\ Jyr = 11.86231\ Eyrs\left(\frac{365.24\ Edays}{1\ Eyr}\right)\left(\frac{24\ Ehrs}{1\ Eday}\right)\left(\frac{1\ Jday}{9.9259\ Ehrs}\right)$$
$$\approx 10476\ Jdays$$

$$P(X=x) = \frac{(x-1)}{10476^x}\left({}_{10476}P_{x-1}\right) \quad \text{for} \quad x \in \{2,3,4,\ldots,10477\}$$

- Both 10476^{25} and ${}_{10476}P_{25}$ cause a TI-84 overflow!

JOVIAN BIRTHDAY AVERAGES

- Mean:

$$E(X) \approx \sqrt{\frac{\pi d}{2}} + \frac{2}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2d}} - \frac{4}{135d} + \cdots \approx 128.947$$

- Median:

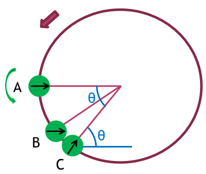
$$n \approx \frac{1+\sqrt{1+4d\ln 4}}{2} \approx 121.017$$

- Mode:

$$x \approx \frac{1+\sqrt{1+4d}}{2} \approx 102.854$$

SIDERIAL VS. SOLAR TIME

- Prograde motion



- For retrograde motion: switch B & C, and

$$\frac{T_{siderial}}{T_{solar}} = \frac{2\pi}{2\pi - \theta}$$

- AB: 1 siderial day
  - Rotate 2π
- AC: 1 solar day
  - Rotate (2π+θ)
  - Revolve θ
- Circle: 1 orbital period
  - Revolve 2π
- Equations:

$$\frac{T_{siderial}}{T_{solar}} = \frac{2\pi}{2\pi + \theta}$$

$$\frac{T_{solar}}{T_{orbital}} = \frac{\theta}{2\pi}$$

USING ASTRONOMICAL DATA

- Astronomical data typically includes the orbital period and the siderial day length, but not the solar day length
- From the equations: $\frac{T_{siderial}}{T_{solar}} = \frac{2\pi}{2\pi \pm \theta}, \quad \frac{T_{solar}}{T_{orbital}} = \frac{\theta}{2\pi}$

  we can substitute for θ to get: $T_{solar} = \dfrac{T_{siderial}}{1 \mp \dfrac{T_{siderial}}{T_{orbital}}}$
- Then:

$$\text{Days in a year} = \frac{T_{orbital}}{T_{solar}} = \frac{T_{orbital}}{T_{siderial}} \mp 1$$

THE PLANETS

Planet	Siderial Day (Edays)	Orbital Period (Edays)	Days in a Year	Birthday Median
Mercury	58.646	87.9691	0.5000	1.47
Venus (retro)	243.019	224.701	1.9246	2.21
Earth	0.99727	365.24	365.24	23.01
Mars	1.025957	686.971	668.59	30.95
Jupiter	0.41354	4332.59	10475.8	121.01
Saturn	0.440417	10759.22	22428.6	184.53
Uranus (retro)	0.71833	30799.095	42877.0	244.30
Neptune	0.6713	60190.03	89660.9	353.06

THE DWARF PLANETS

Dwarf Planet	Siderial Day (Edays)	Orbital Period (Edays)	Days in a Year	Birthday Median
1 Ceres	0.3781	1680.99	4444.9	79.00
134340 Pluto	6.38723	90465.	14162.4	140.62
136108 Haumea	0.163146	103468.	634203.9	938.15
136472 Makemake	0.32379	113183.	349555.8	696.62
136199 Eris	1.079	204624.	189641.3	513.24

CLASSIFICATION

According to the IAU:

- A planet must:
  1. Be in orbit around the sun
  2. Have sufficient mass to have reached hydrostatic equilibrium (a round shape)
  3. Have cleared the neighborhood around its orbit
- If 1,2 only, then “dwarf planet”
- If 1 only, but not a satellite (e.g. moon), then “small solar system body”

OTHER CELESTIAL BODIES

Small Solar System Body (SSSB)	Siderial Day (Edays)	Orbital Period (Edays)	Days in a Year	Birthday Median
2 Pallas (asteroid)	0.32555	1685.87	5177.5	85.22
2060 Chiron (centaur)	0.2466	18539.	75177.	323.3
1P/Halley (comet)	2.2	27500.	12499.	132.1
90377 Sedna (trans-Neptunian object, Oort cloud?)	0.42	4161000.	9907142.	3706.5

Planets do exist around other stars (exoplanets), and orbital period data is often available, but not siderial day lengths. [FUTURE RESEARCH]

THANK YOU!

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- A PDF of the presentation is available at:
  http://www.milefoot.com/about/presentations/ETbirthdays.pdf