# Some Highlights along a Path to Elliptic Curves

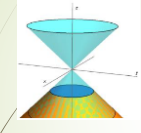
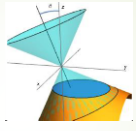
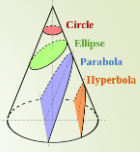
**Part 2: Conic Sections and Rational Points**

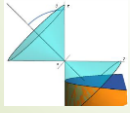
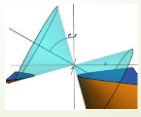
Steven J. Wilson, Fall 2016

## Outline of the Series

1. The World of Algebraic Curves
2. **Conic Sections and Rational Points**
3. Projective Geometry and Bezout's Theorem
4. Solving a Cubic Equation
5. Exploring Cubic Curves
6. Rational Points on Elliptic Curves

## Sections of a Cone

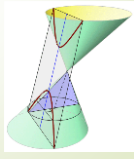



## Early Greek Conic Sections

- Menaechmus was the first to study conics (c350BC)
- Euclid (fl. 300 BC) wrote *Conics*, but it is now lost
- Archimedes (d. c212 BC) wrote: *On Conoids and Spheroids*
  - Work studied volumes of solids of revolution of conic sections
  - Image from: Piero della Francesca (c1416-1492), who illustrated works of Archimedes.

## Apollonius of Perga

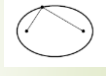
- Lived c262-c190 BC
- *Conics* in 8 "books"
  - Book 8 is lost
  - Image from 9th century Arabic translation
- Euclid probably heavily influenced Books 1-3
- He was the first to use oblique cones
- Mentions foci, never directrix

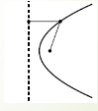


## Conics Defined as Loci

The set of points whose:

- Circle: Distance to center is constant
- Ellipse: Sum of distances to 2 foci is constant
- Parabola: Distance to focus equals distance to directrix
- Hyperbola: Difference of distances to 2 foci is constant





## Algebraic Curves of Degree 2

- The general equation:
$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$
  - John Wallis was first (1655) to define conics by equations.
- Solve for $y$ using the quadratic formula:
$$cy^2 + (bx + e)y + (ax^2 + dx + f) = 0$$
- And assuming $c \neq 0$:
$$y = \frac{-(bx + e) \pm \sqrt{(bx + e)^2 - 4c(ax^2 + dx + f)}}{2c}$$
$$y = \frac{-(bx + e) \pm \sqrt{(b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4cf)}}{2c}$$

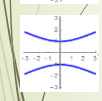
## The quadratic cases:

$$y = \frac{-(bx + e) \pm \sqrt{(b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4cf)}}{2c}$$

Radicand is a quadratic function (parabola):

$$R(x) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4cf)$$

- If $R(x)$ opens down, then $b^2 - 4ac < 0$, and the domain of $R(x)$ is finite, which only happens with an ellipse (or circle).



- If $R(x)$ opens up, then $b^2 - 4ac > 0$, and the domain of $R(x)$ extends to infinity in both directions, which only happens with a hyperbola. [Since $c \neq 0$, parabolas with a vertical axis have been excluded.]

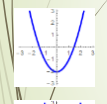
- If $b^2 - 4ac = 0$, then $R(x)$ is a linear function, and the domain extends to infinity in only one direction, which only happens with a parabola.

Some "degenerate" cases are also possible.

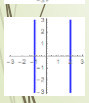
## The other cases:

$$cy^2 + (bx + e)y + (ax^2 + dx + f) = 0$$

If $c = 0$ then $y = \frac{-(ax^2 + dx + f)}{(bx + e)}$

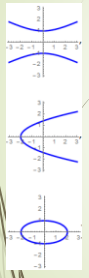


- If $b \neq 0$ then the graph is a hyperbola with a vertical asymptote.
- If $b = 0$ but $e \neq 0$ then the graph is a parabola with a vertical axis.



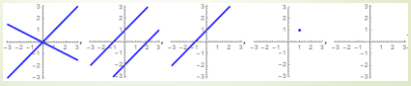
- If $b = e = 0$ then the equation becomes $ax^2 + dx + f = 0$ whose graph is two vertical lines (or one or zero).

## The Discriminant

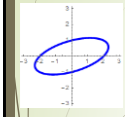


- By defining the discriminant as $D = b^2 - 4ac$ we have:

| Discriminant | Curve | Degenerate Cases |
|---|---|---|
| Positive | Hyperbola | Two intersecting lines |
| Zero | Parabola | Two parallel lines, or One doubled line |
| Negative | Ellipse | A single point, or The empty set |

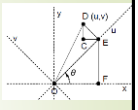


## Rotating the general conic



- The general equation:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$




- To rotate, let:

$$x = u\cos\theta + v\sin\theta, \quad y = -u\sin\theta + v\cos\theta$$

- Then, after a lot of algebra, we find we can "remove" the $bxy$ term by using the substitution $\cot 2\theta = \frac{c-a}{b}$.
  - For $b \neq 0$, since the range of the cotangent function is all real numbers, this always has a solution.
  - For $b = 0$, no rotation is needed.

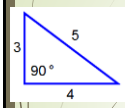
## Diophantine Equations

- A **Diophantine equation** is an equation whose solutions are restricted to the set of integers.
- The Diophantine equation $2x + 3y = 4$ has solutions:

$$\ldots, (-1, 2), (2, 0), (5, -2), \ldots$$

- The Diophantine equation $2x + 2y = 3$ has no solutions:
  - The left side of the equation is even, but the right side is odd.
- In order for the linear Diophantine equation $ax + by = c$ to have solutions, the constant $c$ must be a multiple of the greatest common divisor $\gcd(a, b)$.

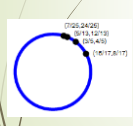
## Integer-Sided Right Triangles




- Can a right triangle have all 3 sides be integers?
  - Yes, the most famous example being 3, 4, 5.
- Sides of right triangles satisfy $a^2 + b^2 = c^2$.
- **Pythagorean Triples** are sets of integers which satisfy the equation $a^2 + b^2 = c^2$.
- Finding Pythagorean Triples is a quadratic Diophantine problem.
- Pythagorean Triples are related to points on the unit circle $x^2 + y^2 = 1$.
  - Use the substitution $x = \frac{a}{c}, \quad y = \frac{b}{c}$
- Those points will be **rational points**. That is, both coordinates of the point will be rational numbers.

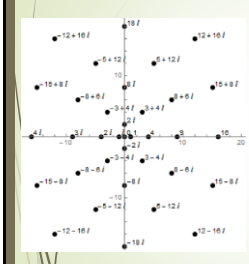
## Pythagorean Triples

- Every primitive Pythagorean Triple can be expressed as $(q^2 - p^2, 2pq, q^2 + p^2)$, where $q > p > 0$ are integers, have no common factors, and exactly one is odd.



| q | p | a | b | c |
|---|---|---|---|---|
| 2 | 1 | 3 | 4 | 5 |
| 3 | 1 | violates "exactly one odd" | | |
| 3 | 2 | 5 | 12 | 13 |
| 4 | 1 | 15 | 8 | 17 |
| 4 | 2 | violates "no common factors" | | |
| 4 | 3 | 7 | 24 | 25 |

## Gaussian Integers



- Every primitive Pythagorean Triple $(q^2 - p^2, 2pq, q^2 + p^2)$ can be produced by squaring a Gaussian integer.
- Let the Gaussian integer be $q + pi$.
- Then $(q + pi)^2 = (q^2 - p^2) + (2pq)i$, which gives the first two values in the Pythagorean Triple.
- And the "Pythagorean Theorem" produces the third:

$$\sqrt{(q^2 - p^2)^2 + (2pq)^2}$$
$$= \sqrt{q^4 + 2p^2q^2 + p^4}$$
$$= q^2 + p^2$$

## Areas of Pythagorean Right Triangles

Some integers are areas of Pythagorean Right Triangles.

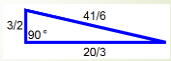
| Triple | 3,4,5 | 5,12,13 | 15,8,17 | 7,24,25 | 6,8,10 |
|---|---|---|---|---|---|
| Area | 6 | 30 | 60 | 84 | 24 |

- The complete list begins: 6, 24, 30, 54, 60, 84, 96, 120, …

Which integers are areas of rational-sided right triangles?

- Such integers are called **congruent numbers**.
- The smallest congruent number is 5. Note that:

$$\left(\frac{3}{2}\right)^2+\left(\frac{20}{3}\right)^2=\left(\frac{41}{6}\right)^2$$




- If denominators were common, the numerators would be the Pythagorean Triple (9, 40, 41).
- Finding congruent numbers efficiently is an unsolved problem.

## Some Rational Point Theorems without Proofs

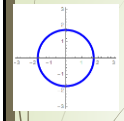
Assuming all given equations have integer coefficients …

- If a line intersects a circle in two rational points, then the slope of the line is a rational number.
- If a line with rational slope intersects a circle twice, and one of the points is a rational point, then the other point is also a rational point.
- If a circle has one rational point, then it has an infinite number of rational points.
- Every rational point on the unit circle has the form

$$\left(\frac{1-m^2}{1+m^2},\frac{2m}{1+m^2}\right)$$

where $m$ is a rational number.

## A Circle With No Rational Points



- The circle $x^2+y^2=3$ has no rational points.

Proof by contradiction.

- Assume $(x,y)$ is a rational point, with coordinates in lowest terms.
- Then integers $p,q,d$ exist with $x=\frac{p}{d},\quad y=\frac{q}{d}$.
- Since we have lowest terms, at least one of $p,q,d$ is odd.
- By substitution, $p^2+q^2=3d^2$.
- If $p$ is even, then $p^2\equiv 0 \bmod 4$. If odd, then $p^2\equiv 1 \bmod 4$.
- Same for $q,d$.
- So $p^2+q^2\equiv 0,1,2 \bmod 4$, and $3d^2\equiv 0,3 \bmod 4$.
- So both sides are $0 \bmod 4$, and all variables are even.
- This contradicts the assumption that the fractions were in lowest terms.

## Integer Sided 60° Triangles

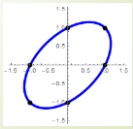
- Can a triangle have a 60° angle and 3 integer sides?
  - Yes, equilateral triangles may. Are there others?
- Sides of the triangle satisfy

$$c^2 = a^2 + b^2 - 2ab\cos 60^\circ$$

$$c^2 = a^2 + b^2 - ab$$

- Triples solving this equation are called **Eisenstein Triples**.

- Do rational points exist on $x^2 + y^2 - xy = 1$ ?
- Yes: (0,1), (1,0), (1,1), etc.
- It will have an infinite number of rational points.



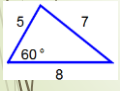
## Eisenstein Triples

- Every rational point on $x^2 + y^2 - xy = 1$ has the form

$$\left(\frac{1-m^2}{1-m+m^2}, \frac{2m-m^2}{1-m+m^2}\right)$$

where $m$ is a rational number.

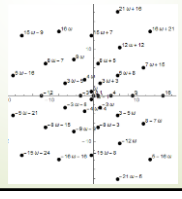
- Every triple $(q^2 - p^2,\ 2pq - p^2,\ p^2 - pq + q^2)$ is an Eisenstein Triple.




| q | p | a | b | c |
|---|---|---|---|---|
| 2 | 1 | 3 | 3 | 3 |
| 3 | 1 | 8 | 5 | 7 |
| 4 | 1 | 15 | 7 | 13 |
| 5 | 1 | 24 | 9 | 21 |
| 5 | 2 | 21 | 16 | 19 |

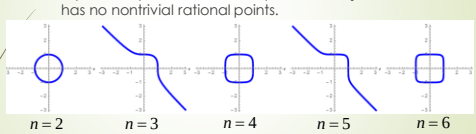
## Eisenstein Integers

- An Eisenstein Integer is a number of the form $q + p\omega$, where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
  - Note: $\omega$ is one of the cube roots of $+1$.
- Every primitive Eisenstein Triple can be produced by squaring an Eisenstein integer.



## Fermat's Last Theorem

- For $n \geq 3$, the equation $a^n + b^n = c^n$ has no solutions where all three variables are positive integers.
- Equivalently, for $n \geq 3$, the equation $x^n + y^n = 1$ has no nontrivial rational points.



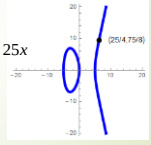
$n=2$ $n=3$ $n=4$ $n=5$ $n=6$

Elliptic Curve

- Andrew Wiles' proof (1994) of Fermat's Last Theorem (1637) investigated properties of elliptic curves.

## Congruent Numbers Again

- Congruent numbers satisfy $a^2 + b^2 = c^2$ and $ab = 2n$, where $a, b, c$ are rational and $n$ is an integer.
- There is a one-to-one correspondence between congruent numbers and the rational points on the curve $y^2 = x^3 - n^2 x$.
- For $n = 5$, we have $y^2 = x^3 - 25x$




- The (unsolved) congruent number problem is related to elliptic curves.

## Challenges:

- Find some more rational points on the unit circle.
- Find some more congruent numbers.
- Can a triangle have a 120° angle and 3 integer sides?