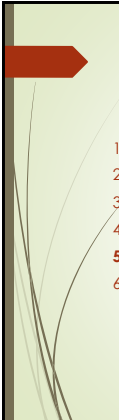



Some Highlights along a Path to Elliptic Curves

Part 5: Exploring Cubic Curves
Steven J. Wilson, Fall 2016




Outline of the Series

1. The World of Algebraic Curves
2. Conic Sections and Rational Points
3. Projective Geometry and Bezout's Theorem
4. Solving a Cubic Equation
- 5. Exploring Cubic Curves**
6. Rational Points on Elliptic Curves



Isaac Newton (1642-1727)



- "If I have seen further it is by standing on the shoulders of giants." - in a letter to Robert Hooke, 1676.

Timeline

- 1545: Cardano published procedures for solving cubic & quartic equations, and how to transform them.
- 1572: Bombelli established rules for negative & imaginary numbers.
- 1591: Viète advocated the use of letters for unknown quantities and coefficients in polynomials.
- 1637: Descartes' *La Geometrie* analyzed geometric problems by using algebraic equations.
- 1655: Wallis began with a general quadratic equation in 2 variables to derive properties of the conic sections.
- c1667: Newton classified the cubic curves.

Newton's Classification



$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0$$

- Newton classified the cubic curves several times.
 - First manuscript in about 1667-8 or 1670.
 - Second manuscript maybe 1678.
 - Systematized in 1695, published in 1704 as an appendix to *Opticks*.
- He found 72 species. He missed 6 in his scheme.
- Features in his classification scheme:
 - Branches: number (1, 2, 3, or 4)
 - Singularities: cusps, nodes, isolated points
 - Asymptotes: number, type, intersections with each other
 - Asymptote-Curve Relationship: intersections, placement
 - Symmetry

Classifying Asymptotes

Type and Number



or "Not asymptotic"

Special Case Intersections



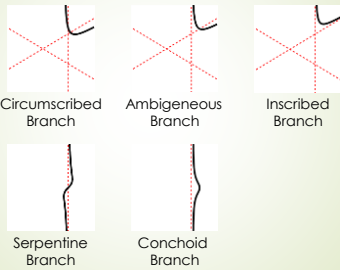
Concurrent

Two Parallel

Line Parallel to Parabolic Axis

How a Curve Intersects the Asymptotes

Asymptote Intersections



Circumscribed Branch

Ambigenuous Branch


Inscribed Branch

Serpentine Branch


Conchoid Branch

How an Asymptote Intersects the Curve


- An asymptote is a **diameter** when its only intersections with the curve are at infinity.



Diameter
Inflection Point



Diameter
Cusp




Not a Diameter
Tangent

- Adiameteral** curves have no asymptotes that are diameters.
- Monodiameteral** curves have one asymptote that is a diameter.
- Tridiameteral** curves have three asymptotes that are diameters.

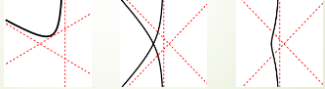
The Asymptotic Triangle

- The relationship of the curve to the asymptotic triangle can occur in different ways.

At Vertex



At Base

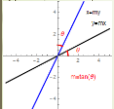


The General Cubic Curve

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0$$

Every cubic curve has at least one point at infinity.

- Homogenize, then let $z = 0$ to find points at infinity.
 $Ax^3 + Bx^2y + Cxy^2 + Dy^3 = 0$
 - If $A = 0$, then $(1, 0, 0)$ is a point at infinity.
 - If $A \neq 0$ and $y = 0$, then $x = 0$, but $(0, 0, 0)$ is not a point.
 - So $y \neq 0$, and $A\left(\frac{x}{y}\right)^3 + B\left(\frac{x}{y}\right)^2 + C\left(\frac{x}{y}\right) + D = 0$
 - This equation has at least one real solution $\frac{x}{y} = m$.



Rotate to move that point at infinity to the y-axis: $(0, 1, 0)$
 Then $D = 0$: every vertical line intersects only 2 finite points.
 $(Cx + Gy)^2 + (Bx^2 + Fx + I)y + (Ax^3 + Ex^2 + Hx + J) = 0$

Newton's 4 Cases

$$(Cx + G)y^2 + (Bx^2 + Fx + I)y + (Ax^3 + Ex^2 + Hx + J) = 0$$

After making several more transformations (horizontal, vertical, and/or shear), Newton was able to classify all cubic curves into 4 cases.

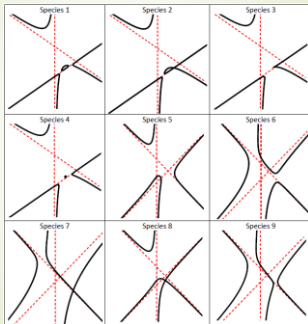
Case	Conditions	Equation	Number of Species
1	$C \neq 0$	$xy^2 + ey = ax^3 + bx^2 + cx + d$	71
2	$C = G = 0, F \neq 0$	$xy = ax^3 + bx^2 + cx + d$	1
3	$C = 0, G \neq 0$	$y^2 = ax^3 + bx^2 + cx + d$	5
4	$C = F = G = 0$	$y = ax^3 + bx^2 + cx + d$	1

Note: When rotating to remove the Dy^3 term, Newton chose to avoid using an asymptote of multiplicity 2. That choice meant the condition $C = 0$ would also imply $B = 0$.

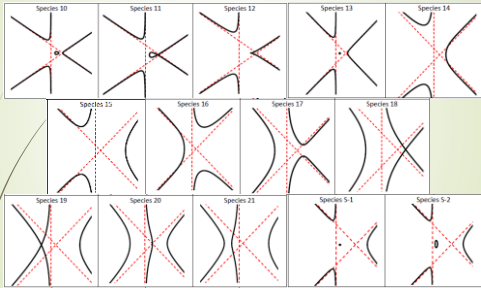
Newton's Case 1: $xy^2 + ey = ax^3 + bx^2 + cx + d$

- Class 1: $a > 0$, with three linear asymptotes
 - Adiametral: $e \neq 0$ 9 species
 - Monodiametral: $e = 0, b^2 - 4ac \neq 0$ 14 species
 - Tridiametral: $e = 0, b^2 - 4ac = 0$ 4 species
 - Concurrent asymptotes: $b = 0$ 9 species
- Class 2: $a < 0$, with one linear asymptote
 - Adiametral: $e \neq 0$ 6 species
 - Monodiametral: $e = 0$ 7 species
- Class 3: $a = 0, b \neq 0$, with one linear & one parabolic asymptote
 - Adiametral: $e \neq 0$ 7 species
 - Monodiametral: $e = 0$ 6 species
- Class 4: $a = b = 0$
 - Three linear asymptotes (2 parallel): $c > 0$ 4 species
 - One linear asymptote: $c < 0$ 3 species
 - Two linear asymptotes (1 repeated): $c = 0$ 2 species

Case 1, Class 1, Adiametral

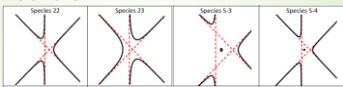


Case 1, Class 1, Monodiametral

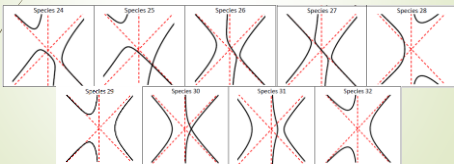


The Rest of Case 1, Class 1

Tridiametral

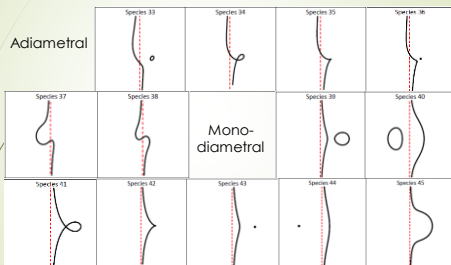


Concurrent Asymptotes



Case 1, Class 2

Adiametral



Case 1, Class 3

Adidamtral

Species 46	Species 47	Species 48	Species 49
Species 50	Species 51	Species 52	Species 53
Species 54	Species 55	Species 56	Species 5-5
			Species 5-4

Mono-diamtral

Case 1, Class 4

3 Linear Asymptotes (2 parallel)

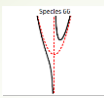
Species 57	Species 58	Species 59	Species 60
Species 61	Species 62	Species 63	
Species 64	Species 65		

1 Linear Asymptote


2 Linear Asymptotes (1 repeated)

Cases 2, 3, and 4

Case 2 Newton's Trident



Case 4 Cubic Polynomial Function



Case 3 $y^2 = ax^3 + bx^2 + cx + d$

Species 67	Species 68	Species 69	Species 70	Species 72
------------	------------	------------	------------	------------

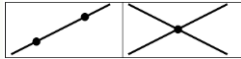
Counting Singularities

- An irreducible algebraic curve of degree n has at most $\frac{(n-1)(n-2)}{2}$ singularities.
- An algebraic curve of degree n with no repeated components can have at most $\frac{n(n-1)}{2}$ singularities.

Degree	Max Number of Singularities when Irreducible	Max Number of Singularities with No Repeated Components
1	0	0
2	0	1
3	1	3
4	3	6
5	6	10

Duality

- A principle in projective geometry where a theorem (or axiom) involving points and lines is also true if the terms "points" and "lines" are interchanged.
 - Two points determine one line.
 - Two lines determine one point.

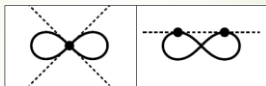


- Given a curve C and an arbitrary line, the **degree** is the number of points of C on that line.
- Given a curve C and an arbitrary point, the **class** is the number of tangent lines of C containing that point.



Duals of Singularities

- A **node** is a point at which two lines are tangent.
- A **bitangent** is a line which is tangent at two points.



- A **cusp** occurs when a point moving along the curve reverses direction.
- An **inflection point** occurs when a tangent line moving along a curve reverses direction.



Plücker Formulas



- In 1834, Plücker observed the following relations, for irreducible curves of degree at least 2:
 $class = (degree)(degree - 1) - 2(bitangents) - 3(cusps)$
 $degree = (class)(class - 1) - 2(bitangents) - 3(inflexion\ points)$
 $inflexion\ points = 3(degree)(degree - 2) - 6(nodes) - 8(cusps)$
 $cusps = 3(class)(class - 2) - 6(bitangents) - 8(inflexion\ points)$
- Only 3 of these equations are independent, but they are sufficient to find all 6 quantities.
- Additionally, we can define the **genus** of a curve:

$$g = \frac{(n-1)(n-2)}{2} - \sum \delta_s$$

where δ_s is the delta-invariant of the singularity s .

- For simple nodes and cusps, $\delta_s = 1$.
- Because curves are really 2D surfaces in a 4D space, the genus corresponds to the "number of holes" in the surface (as in topology).

Some Plücker Numbers

Degree	Nodes	Cusps	Class	Bitangents	Inflexion Pts	Genus
2	0	0	2	0	0	0
3	1	0	4	0	3	0
3	0	1	3	0	1	0
3	0	0	6	0	9	1
4	3	0	6	4	6	0
4	2	1	5	2	4	0
4	1	2	4	1	2	0
4	0	3	3	1	0	0
4	2	0	8	8	12	1
4	1	1	7	4	10	1
4	0	2	6	1	8	1
4	1	0	10	16	18	2
4	0	1	9	10	16	2
4	0	0	12	28	24	3

Challenges

- Identify all of the elliptic curves among Newton's 78 species. How many are there?
- Find a degree 4 algebraic curve for each of the 10 types enumerated by Plücker.
- Some References (both available online):
 - Ball, W.W. Rouse, "On Newton's Classification of Cubic Curves", Proceedings of the London Mathematical Society, vol. 22 (1891), p. 104-143.
 - Talbot, C.R.M., "Sir Isaac Newton's Enumeration of Lines of the Third Order", 1860.
