# Some Highlights along a Path to Elliptic Curves

**Part 5: Exploring Cubic Curves**

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## Outline of the Series

1. The World of Algebraic Curves
2. Conic Sections and Rational Points
3. Projective Geometry and Bezout's Theorem
4. Solving a Cubic Equation
5. **Exploring Cubic Curves**
6. Rational Points on Elliptic Curves

## Isaac Newton (1642-1727)

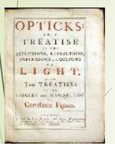
- "If I have seen further it is by standing on the shoulders of giants." - in a letter to Robert Hooke, 1676.

Timeline

- 1545: Cardano published procedures for solving cubic & quartic equations, and how to transform them.
- 1572: Bombelli established rules for negative & imaginary numbers.
- 1591: Viete advocated the use of letters for unknown quantities and coefficients in polynomials.
- 1637: Descartes' *La Geometrie* analyzed geometric problems by using algebraic equations.
- 1655: Wallis began with a general quadratic equation in 2 variables to derive properties of the conic sections.
- c1667: Newton classified the cubic curves.

## Newton's Classification

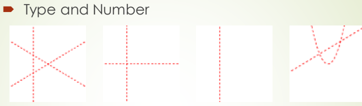
$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0$$




- Newton classified the cubic curves several times.
  - First manuscript in about 1667-8 or 1670.
  - Second manuscript maybe 1678.
  - Systematized in 1695, published in 1704 as an appendix to *Opticks*.
- He found 72 species. He missed 6 in his scheme.
- Features in his classification scheme:
  - Branches: number (1, 2, 3, or 4)
  - Singularities: cusps, nodes, isolated points
  - Asymptotes: number, type, intersections with each other
  - Asymptote-Curve Relationship: intersections, placement
  - Symmetry

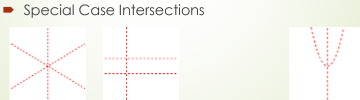
## Classifying Asymptotes

- Type and Number



or "Not asymptotic"

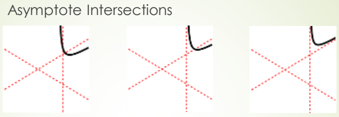
- Special Case Intersections



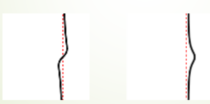
Concurrent Two Parallel Line Parallel to Parabolic Axis

## How a Curve Intersects the Asymptotes

- Asymptote Intersections



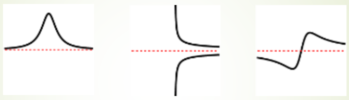
Circumscribed Branch Ambigeneous Branch Inscribed Branch



Serpentine Branch Conchoid Branch

## How an Asymptote Intersects the Curve

- An asymptote is a **diameter** when its only intersections with the curve are at infinity.



Diameter Inflection Point | Diameter Cusp | Not a Diameter Tangent

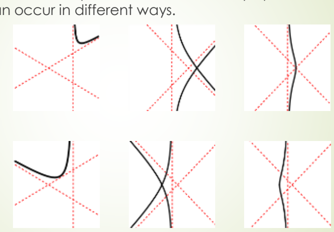
- **Adiametral** curves have no asymptotes that are diameters.
- **Monodiametral** curves have one asymptote that is a diameter.
- **Tridiametral** curves have three asymptotes that are diameters.

## The Asymptotic Triangle

- The relationship of the curve to the asymptotic triangle can occur in different ways.

At Vertex

At Base



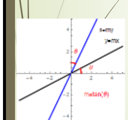
## The General Cubic Curve

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0$$

Every cubic curve has at least one point at infinity.

- Homogenize, then let $z = 0$ to find points at infinity.

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 = 0$$



- If $A = 0$, then $(1,0,0)$ is a point at infinity.
- If $A \neq 0$ and $y = 0$, then $x = 0$, but $(0,0,0)$ is not a point.
- So $y \neq 0$, and $A\left(\frac{x}{y}\right)^3 + B\left(\frac{x}{y}\right)^2 + C\left(\frac{x}{y}\right) + D = 0$
- This equation has at least one real solution $\frac{x}{y} = m$.

Rotate to move that point at infinity to the y-axis: $(0,1,0)$

Then $D = 0$: every vertical line intersects only 2 finite points.

$$(Cx + G)y^2 + (Bx^2 + Fx + I)y + (Ax^3 + Ex^2 + Hx + J) = 0$$

## Newton's 4 Cases

$$(Cx+G)y^2+(Bx^2+Fx+I)y+(Ax^3+Ex^2+Hx+J)=0$$

After making several more transformations (horizontal, vertical, and/or shear), Newton was able to classify all cubic curves into 4 cases.

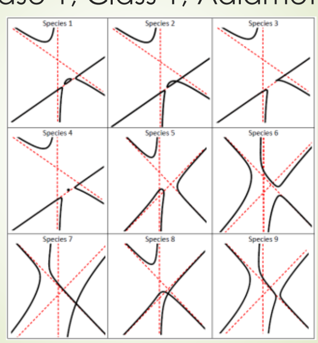
| Case | Conditions | Equation | Number of Species |
|---|---|---|---|
| 1 | $C \neq 0$ | $xy^2+ey=ax^3+bx^2+cx+d$ | 71 |
| 2 | $C=G=0, F\neq 0$ | $xy=ax^3+bx^2+cx+d$ | 1 |
| 3 | $C=0, G\neq 0$ | $y^2=ax^3+bx^2+cx+d$ | 5 |
| 4 | $C=F=G=0$ | $y=ax^3+bx^2+cx+d$ | 1 |

Note: When rotating to remove the $Dy^3$ term, Newton chose to avoid using an asymptote of multiplicity 2. That choice meant the condition $C=0$ would also imply $B=0$.

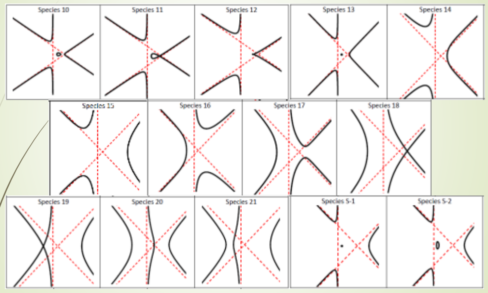
## Newton's Case 1: $xy^2+ey=ax^3+bx^2+cx+d$

- Class 1: $a>0$, with three linear asymptotes
  - Adiametral: $e\neq 0$ 9 species
  - Monodiametral: $e=0, b^2-4ac\neq 0$ 14 species
  - Tridiametral: $e=0, b^2-4ac=0$ 4 species
  - Concurrent asymptotes: $b=0$ 9 species
- Class 2: $a<0$, with one linear asymptote
  - Adiametral: $e\neq 0$ 6 species
  - Monodiametral: $e=0$ 7 species
- Class 3: $a=0, b\neq 0$, with one linear & one parabolic asymptote
  - Adiametral: $e\neq 0$ 7 species
  - Monodiametral: $e=0$ 6 species
- Class 4: $a=b=0$
  - Three linear asymptotes (2 parallel): $c>0$ 4 species
  - One linear asymptote: $c<0$ 3 species
  - Two linear asymptotes (1 repeated): $c=0$ 2 species

## Case 1, Class 1, Adiametral

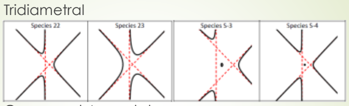


## Case 1, Class 1, Monodiametral

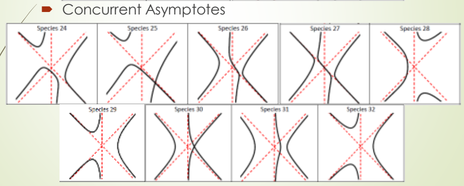



## The Rest of Case 1, Class 1

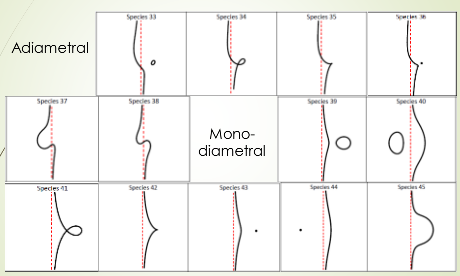
- Tridiametral




- Concurrent Asymptotes

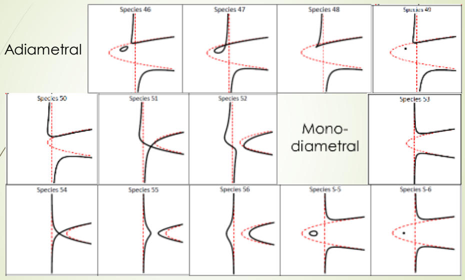



## Case 1, Class 2



## Case 1, Class 3

Adiametral

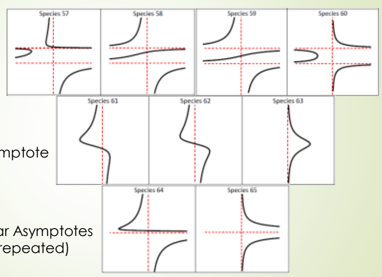



## Case 1, Class 4

3 Linear Asymptotes
(2 parallel)

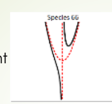
1 Linear Asymptote

2 Linear Asymptotes
(1 repeated)

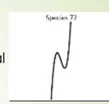


## Cases 2, 3, and 4

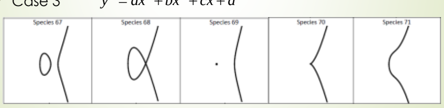
Case 2
Newton's Trident



Case 4
Cubic Polynomial
Function



Case 3 $y^2 = ax^3 + bx^2 + cx + d$



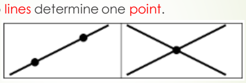
## Counting Singularities

- An irreducible algebraic curve of degree $n$ has at most $\frac{(n-1)(n-2)}{2}$ singularities.
- An algebraic curve of degree $n$ with no repeated components can have at most $\frac{n(n-1)}{2}$ singularities.

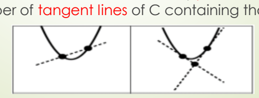
| Degree | Max Number of Singularities when Irreducible | Max Number of Singularities with No Repeated Components |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0 | 1 |
| 3 | 1 | 3 |
| 4 | 3 | 6 |
| 5 | 6 | 10 |

## Duality

- A principle in projective geometry where a theorem (or axiom) involving points and lines is also true if the terms "points" and "lines" are interchanged.
  - Two points determine one line.
  - Two lines determine one point.

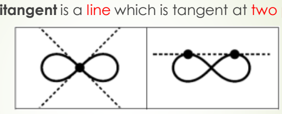


- Given a curve C and an arbitrary line, the **degree** is the number of points of C on that line.
- Given a curve C and an arbitrary point, the **class** is the number of tangent lines of C containing that point.

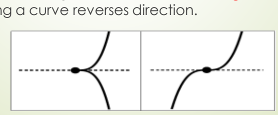


## Duals of Singularities

- A **node** is a point at which two lines are tangent.
- A **bitangent** is a line which is tangent at two points.



- A **cusp** occurs when a point moving along the curve reverses direction.
- An **inflection point** occurs when a tangent line moving along a curve reverses direction.



## Plücker Formulas

- In 1834, Plücker observed the following relations, for irreducible curves of degree at least 2:

$$\text{class} = (\text{degree})(\text{degree}-1) - 2(\text{nodes}) - 3(\text{cusps})$$
$$\text{degree} = (\text{class})(\text{class}-1) - 2(\text{bitangents}) - 3(\text{inflection points})$$
$$\text{inflection points} = 3(\text{degree})(\text{degree}-2) - 6(\text{nodes}) - 8(\text{cusps})$$
$$\text{cusps} = 3(\text{class})(\text{class}-2) - 6(\text{bitangents}) - 8(\text{inflection points})$$

- Only 3 of these equations are independent, but they are sufficient to find all 6 quantities.
- Additionally, we can define the **genus** of a curve:

$$g = \frac{(n-1)(n-2)}{2} - \sum \delta_s$$

  where $\delta_s$ is the delta-invariant of the singularity $s$.
  - For simple nodes and cusps, $\delta_s = 1$.
- Because curves are really 2D surfaces in a 4D space, the genus corresponds to the "number of holes" in the surface (as in topology).

## Some Plücker Numbers

| Degree | Nodes | Cusps | Class | Bitangents | Inflection Pts | Genus |
|---|---|---|---|---|---|---|
| 2 | 0 | 0 | 2 | 0 | 0 | 0 |
| 3 | 1 | 0 | 4 | 0 | 3 | 0 |
| 3 | 0 | 1 | 3 | 0 | 1 | 0 |
| 3 | 0 | 0 | 6 | 0 | 9 | 1 |
| 4 | 3 | 0 | 6 | 4 | 6 | 0 |
| 4 | 2 | 1 | 5 | 2 | 4 | 0 |
| 4 | 1 | 2 | 4 | 1 | 2 | 0 |
| 4 | 0 | 3 | 3 | 1 | 0 | 0 |
| 4 | 2 | 0 | 8 | 8 | 12 | 1 |
| 4 | 1 | 1 | 7 | 4 | 10 | 1 |
| 4 | 0 | 2 | 6 | 1 | 8 | 1 |
| 4 | 1 | 0 | 10 | 16 | 18 | 2 |
| 4 | 0 | 1 | 9 | 10 | 16 | 2 |
| 4 | 0 | 0 | 12 | 28 | 24 | 3 |

## Challenges

- Identify all of the elliptic curves among Newton's 78 species. How many are there?
- Find a degree 4 algebraic curve for each of the 10 types enumerated by Plücker.

- Some References (both available online):


  - Ball, W.W. Rouse, "On Newton's Classification of Cubic Curves", Proceedings of the London Mathematical Society, vol. 22 (1891), p. 104-143.
  - Talbot, C.R.M., "Sir Isaac Newton's Enumeration of Lines of the Third Order", 1860.