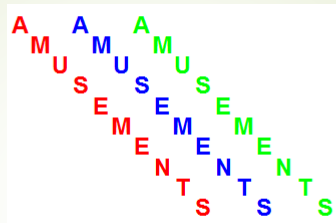

Parallel Amusements

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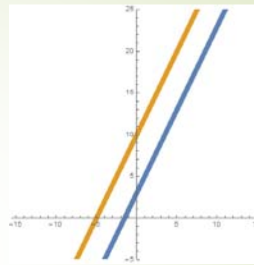
KAMATYC, Wichita, March 2016

Basic Facts

- Definition: Two lines are **parallel** if …
- … they lie in the same plane and do not intersect.

- Parallel lines have equal slopes.
- Two distinct lines with the same slope are parallel.

Parallel Lines



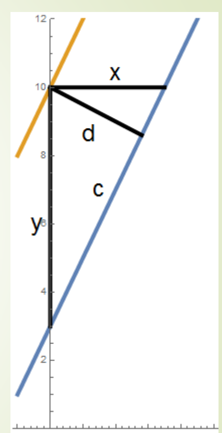
- Consider the lines:

$$y = 2x + 3$$
$$y = 2x + 10$$

- First person to answer this question gets a prize:
- How far apart are they?

- SURPRISE: The answer is NOT 7.

Distance between lines



Similar Triangles: $\frac{y}{c} = \frac{x}{d}$

Thus:

$$c = \left(\frac{y}{x}\right)d$$

$$\sqrt{y^2 - d^2} = \left(\frac{y}{x}\right)d$$

But $\frac{y}{x} = 2$ and $y = 7$, therefore:

$$\sqrt{49 - d^2} = 2d$$
$$49 - d^2 = 4d^2$$
$$49 = 5d^2$$
$$d = \frac{7}{\sqrt{5}} \approx 3.13$$

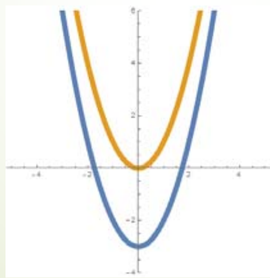
Parallel Lines are equidistant.

Parallel Parabolas?

- Are the parabolas

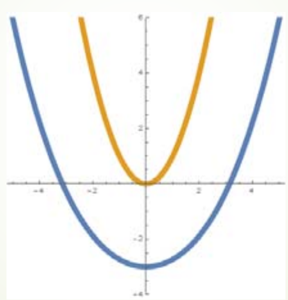
$$y = x^2$$

$$y = x^2 + 3$$ equidistant?

-  No.

Parallel Curves?

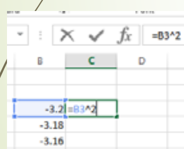
- How do we produce a curve that is parallel (equidistant) to the parabola?



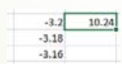
- With Excel, we can avoid using calculus.
- But calculus will help us understand.

Creating an Excel Graph

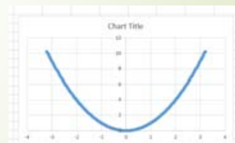
- Column B: enter x-values, say
  -3.2, -3.18, -3.16, …, +3.2
- Column C: enter first y-value, using a formula, then select cell, hover over lower right corner, double-click
- Graph: put cursor in table, Ctrl-A, Insert, Chart

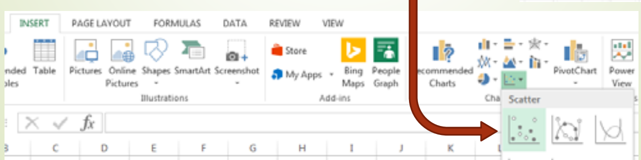


Hover here



Pick this chart

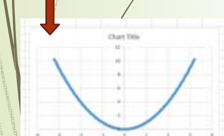




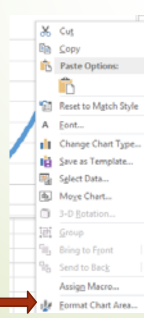
Improving the Excel Graph

- To control scales and max/min:
  - Right-click, choose Format Chart Area.
  - Chart Options, choose square, Size, enter height and width, lock aspect ratio.

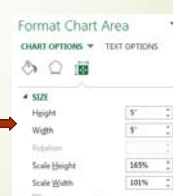
Click here



Then choose this



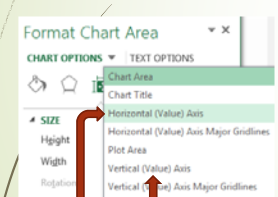
And you get this



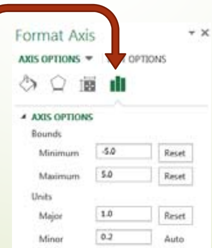
Wait, there's more!

Improving the Excel Graph

- Chart Options drop-down menu, choose an Axis.
- Bars, Axis Options, enter bounds (not Auto).
- Repeat for the other axis.

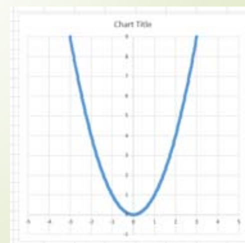


Two axes



Choose bars

Now we have a squared-up graph



Parallel Curve Math

- Get slope of tangent line (or close secant line), then slope of normal

$$m_{\tan} = \frac{y_2 - y_1}{x_2 - x_1} \qquad m_{\text{normal}} = \frac{-1}{m_{\tan}}$$

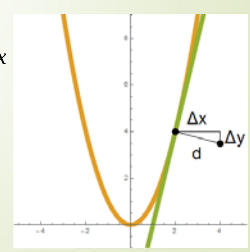
- Given distance d, decompose into dx and dy

$$d^2 = (\Delta x)^2 + (\Delta y)^2 \quad \text{and} \quad m_{\text{normal}} = \frac{\Delta y}{\Delta x}$$

$$\Rightarrow \quad \Delta x = \frac{d}{\sqrt{1 + m_{\text{normal}}^2}} \quad \text{and} \quad \Delta y = m_{\text{normal}} \Delta x$$

- Describe new point

$$x_{new} = x + \Delta x \quad \text{and} \quad y_{new} = y + \Delta y$$



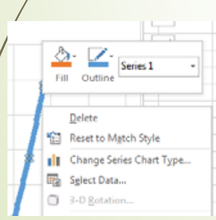
Parallel Curve Formulas in Excel

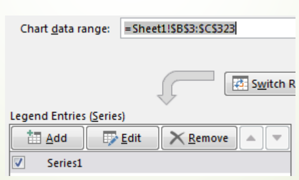
- Slope: =(C5-C3)/(B5-B3)
- Normal slope: =-1/D4
- Delta-x (for d=3): = 3/SQRT(1+E4^2)
- Delta-y: =E4*F4
- New x: =B4+F4
- New y: =C4+G4
- Copy all columns down.

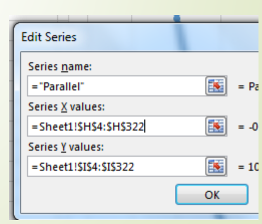
x	y	m-tan	m-norm	delta-x	delta-y	new-x	new-y
-3.2	10.24						
-3.18	10.1124	-6.36	0.157233	2.963591	0.465973	-0.21641	10.57837
-3.16	9.9856	-6.32	0.158228	2.963137	0.468851	-0.19686	10.45445

Parallel Curve Graph in Excel

- Select data
- Note original series values, then select Add
- Enter new values (one less row on each end)
- If needed, adjust axes values of graph

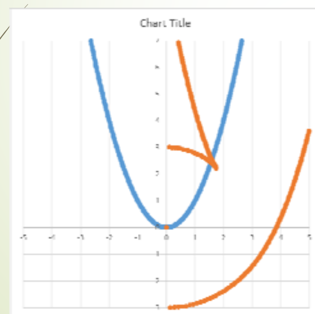


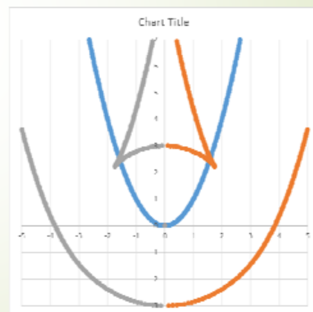




First Attempt at a Graph

- SURPRISE: Not quite what we expected.
- Need 2 new columns: $x_{new} = x - \Delta x$ and $y_{new} = y - \Delta y$





Is this a polynomial function?

- No, it does not pass the vertical line test.
- Even if it did, polynomials do not have cusps.

- But yes (SURPRISE), sort of…
- If you think of the curve as $f(x, y) = 0$, that is, as
- A level curve of a bivariate polynomial function.

Parametric Form

- Beginning with $y = f(x)$, which is $(t, f(t))$
- Recall $\Delta x = \dfrac{d}{\sqrt{1+m_{\text{normal}}^2}}$ and $\Delta y = m_{\text{normal}} \Delta x$
- But $m_{\text{normal}} = \dfrac{-1}{f'(x)}$
- So $(x+\Delta x, y+\Delta y) = \left(t + \dfrac{d\ f'(t)}{\sqrt{1+\left(f'(t)\right)^2}},\quad f(t) - \dfrac{d}{\sqrt{1+\left(f'(t)\right)^2}} \right)$

Parametric Polynomial System

- Rewrite new $(x, y) = \left(t + \dfrac{d\ f'(t)}{\sqrt{1+\left(f'(t)\right)^2}},\quad f(t) - \dfrac{d}{\sqrt{1+\left(f'(t)\right)^2}} \right)$
- As a system $x = t + \dfrac{d\ f'(t)}{\sqrt{1+\left(f'(t)\right)^2}}$ and $y = f(t) - \dfrac{d}{\sqrt{1+\left(f'(t)\right)^2}}$
- Isolate radical terms and square both sides:

$$(x-t)\sqrt{1+\left(f'(t)\right)^2} = d\ f'(t) \quad \text{and} \quad (y-f(t))\sqrt{1+\left(f'(t)\right)^2} = -d$$

$$(x-t)^2\left(1+\left(f'(t)\right)^2\right) = d^2\left(f'(t)\right)^2 \quad \text{and} \quad (y-f(t))^2\left(1+\left(f'(t)\right)^2\right) = d^2$$

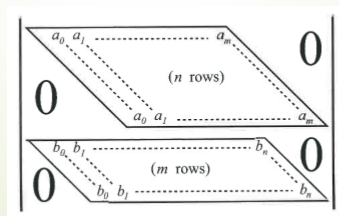
$$\begin{cases} (x-t)^2\left(1+\left(f'(t)\right)^2\right) - d^2\left(f'(t)\right)^2 = 0 \\ (y-f(t))^2\left(1+\left(f'(t)\right)^2\right) - d^2 = 0 \end{cases}$$

Bivariate Polynomial Form

- Any parametric polynomial system

$$\begin{cases} a_0 t^m + a_1 t^{m-1} + \ldots + a_m = 0 \\ b_0 t^n + b_1 t^{n-1} + \ldots + b_n = 0 \end{cases}$$

- can be transformed into a bivariate polynomial form by using the resultant, which is the determinant:



Our Parallel Curve, Parametrically

$$f(x) = x^2, \quad \text{with} \quad d = 3$$

- Original curve, parametric: (t, t^2)
- Slope is $f'(x) = 2x$
- Parallel curve, parametric: $\left(t + \frac{3(2t)}{\sqrt{1+(2t)^2}}, \; t^2 - \frac{3}{\sqrt{1+(2t)^2}} \right)$
- System: $x = t + \frac{6t}{\sqrt{1+4t^2}}$ and $y = t^2 - \frac{3}{\sqrt{1+4t^2}}$

$$(x-t)^2(1+4t^2) - 36t^2 = 0 \quad \text{and} \quad (y-t^2)^2(1+4t^2) - 9 = 0$$

$$\begin{cases} 4t^4 - 8xt^3 + (4x^2 - 35)t^2 - 2xt + x^2 = 0 \\ 4t^6 + (1-8y)t^4 + (4y^2 - 2y)t^2 + (y^2 - 9) = 0 \end{cases}$$

Our Parallel Curve's Resultant

- System: $\begin{cases} 4t^4 - 8xt^3 + \left(4x^2 - 35\right)t^2 - 2xt + x^2 = 0 \\ 4t^6 + \left(1 - 8y\right)t^4 + \left(4y^2 - 2y\right)t^2 + \left(y^2 - 9\right) = 0 \end{cases}$
- Resultant:

$$\det \begin{bmatrix} 4 & -8x & (4x^2-35) & -2x & x^2 & & & & & \\ & 4 & -8x & (4x^2-35) & -2x & x^2 & & & & \\ & & 4 & -8x & (4x^2-35) & -2x & x^2 & & & \\ & & & 4 & -8x & (4x^2-35) & -2x & x^2 & & \\ & & & & 4 & -8x & (4x^2-35) & -2x & x^2 & \\ & & & & & 4 & -8x & (4x^2-35) & -2x & x^2 \\ 4 & 0 & (1-8y) & 0 & (4y^2-2y) & 0 & (y^2-9) & & & \\ & 4 & 0 & (1-8y) & 0 & (4y^2-2y) & 0 & (y^2-9) & & \\ & & 4 & 0 & (1-8y) & 0 & (4y^2-2y) & 0 & (y^2-9) & \\ & & & 4 & 0 & (1-8y) & 0 & (4y^2-2y) & 0 & (y^2-9) \end{bmatrix}$$

Our Parallel Curve's Bivariate Polynomial

- With the help of Mathematica

```
In[4]:= Resultant[ (x - t) ^2 (1 + 4 t^2) - 36 t^2, (y - t^2) ^2 (1 + 4 t^2) - 9, t]
```

Out[4]=

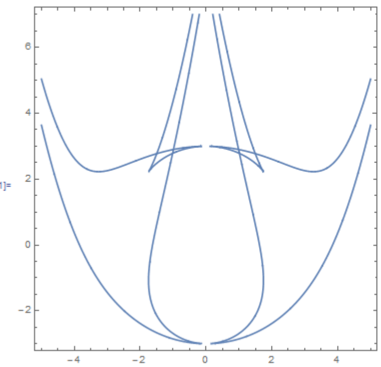
$$2\,504\,040\,397\,056 - 1\,958\,472\,566\,784\, x^2 + 558\,274\,125\,312\, x^4 - 82\,011\,543\,552\, x^6 + 6\,575\,323\,392\, x^8 - 283\,336\,704\, x^{10} + 5\,308\,416\, x^{12} - 1\,148\,454\,236\,160\, y + 488\,761\,537\,536\, x^2 y - 45\,575\,073\,792\, x^4 y - 1\,377\,285\,120\, x^6 y + 367\,939\,584\, x^8 y - 15\,925\,248\, x^{10} y - 359\,145\,570\,816\, y^2 + 316\,760\,647\,680\, x^2 y^2 - 78\,921\,423\,360\, x^4 y^2 + 8\,802\,680\,832\, x^6 y^2 - 467\,804\,160\, x^8 y^2 + 10\,616\,832\, x^{10} y^2 + 240\,162\,693\,120\, y^3 - 69\,427\,860\,480\, x^2 y^3 + 828\,776\,448\, x^4 y^3 + 634\,355\,712\, x^6 y^3 - 37\,158\,912\, x^8 y^3 - 12\,502\,128\,384\, y^4 - 12\,641\,329\,152\, x^2 y^4 + 2\,512\,207\,872\, x^4 y^4 - 148\,635\,648\, x^6 y^4 + 5\,308\,416\, x^8 y^4 - 10\,834\,145\,280\, y^5 + 1\,871\,216\,640\, x^2 y^5 + 180\,486\,144\, x^4 y^5 - 21\,233\,664\, x^6 y^5 + 2\,340\,347\,904\, y^6 + 180\,486\,144\, x^2 y^6 + 31\,850\,496\, x^4 y^6 - 185\,794\,560\, y^7 - 21\,233\,664\, x^2 y^7 + 5\,308\,416\, y^8$$

- A 12th degree polynomial
- 39 terms

Graph of the Resultant

In[11]:=
```
ContourPlot[ %4 == 0, {x, -5, 5}, {y, -3, 7}, PlotPoints → 500]
```

Out[11]=



- Another SURPRISE: … 4 branches, not 2
- We squared both sides, extraneous solutions

Factoring the Resultant

- Factoring

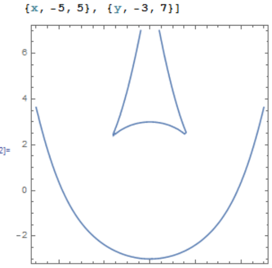
In[19]:=
```
Factor[ %4]
```

Out[19]= $20736\,(-12321 + 3708\,x^2 - 431\,x^4 + 16\,x^6 + 2664\,y + 70\,x^2\,y - 40\,x^4\,y + 1225\,y^2 - 256\,x^2\,y^2 + 16\,x^4\,y^2 - 296\,y^3 - 32\,x^2\,y^3 + 16\,y^4)\,(-9801 + 4716\,x^2 - 423\,x^4 + 16\,x^6 + 2376\,y - 234\,x^2\,y - 8\,x^4\,y + 945\,y^2 - 320\,x^2\,y^2 + 16\,x^4\,y^2 - 264\,y^3 - 32\,x^2\,y^3 + 16\,y^4)$

- Graphing Each Factor (6th degree, 13 terms)

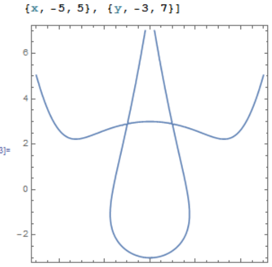
In[22]:=
```
ContourPlot[Evaluate[%13[[2]] == 0],
  {x, -5, 5}, {y, -3, 7}]
```

Out[22]=



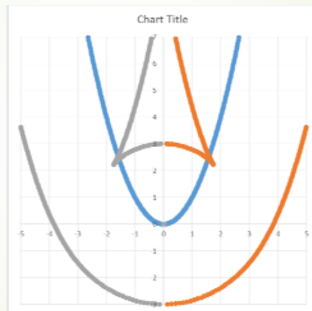
In[23]:=
```
ContourPlot[Evaluate[%13[[3]] == 0],
  {x, -5, 5}, {y, -3, 7}]
```

Out[23]=



The Answer Page

- The Curve Parallel to $y = x^2$, at a distance of 3, is:



$$16x^6 + 16x^4y^2 - 40x^4y - 32x^2y^3 - 431x^4 - 256x^2y^2 + 16y^4$$
$$+70x^2y - 296y^3 + 3708x^2 + 1225y^2 + 2664y - 12321 = 0$$