

# Parallel Amusements

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## Basic Facts

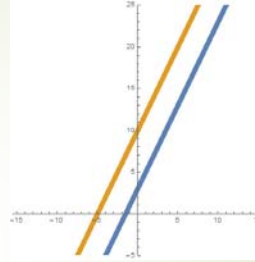
- Definition: Two lines are **parallel** if ...
- ... they lie in the same plane and do not intersect.
- Parallel lines have equal slopes.
- Two distinct lines with the same slope are parallel.

## Parallel Lines

- Consider the lines:

$$y = 2x + 3$$

$$y = 2x + 10$$



- First person to answer this question gets a prize:
- How far apart are they?
- SURPRISE:** The answer is NOT 7.

## Distance between lines

Similar Triangles:  $\frac{y}{c} = \frac{x}{d}$

Thus:

$$c = \left(\frac{y}{x}\right)d$$

$$\sqrt{y^2 - d^2} = \left(\frac{y}{x}\right)d$$

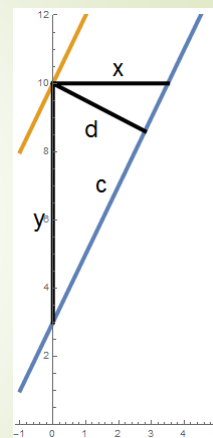
But  $\frac{y}{x} = 2$  and  $y = 7$ , therefore:

$$\sqrt{49 - d^2} = 2d$$

$$49 - d^2 = 4d^2$$

$$49 = 5d^2$$

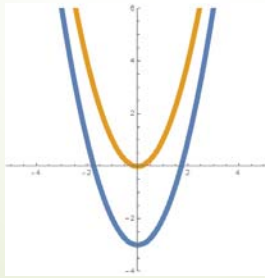
$$d = \frac{7}{\sqrt{5}} \approx 3.13$$



Parallel Lines are equidistant.

## Parallel Parabolas?

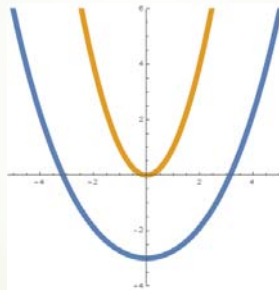
- Are the parabolas  
 $y = x^2$   
 $y = x^2 + 3$  equidistant?



No.

## Parallel Curves?

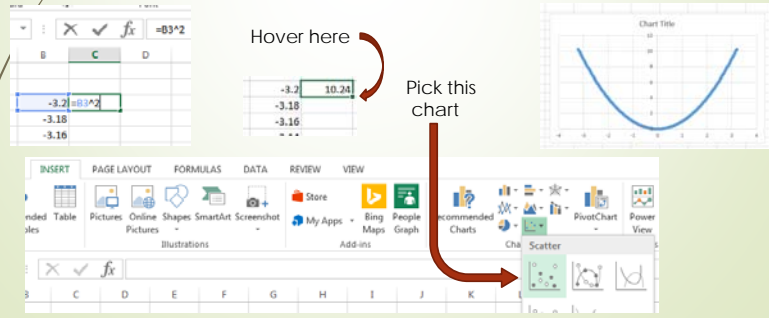
- How do we produce a curve that is parallel (equidistant) to the parabola?



- With Excel, we can avoid using calculus.
- But calculus will help us understand.

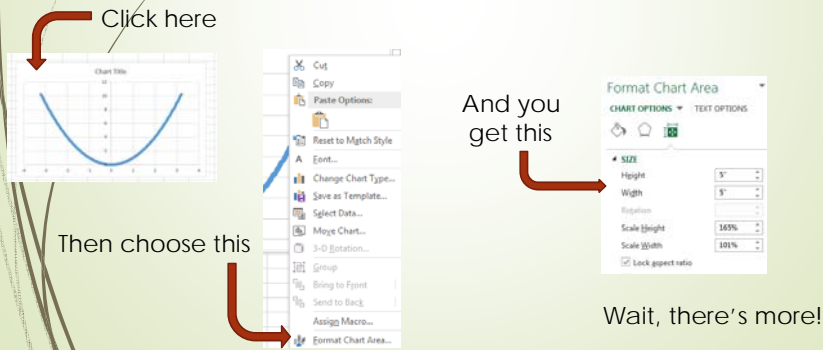
## Creating an Excel Graph

- ▶ Column B: enter x-values, say -3.2, -3.18, -3.16, ..., +3.2
- ▶ Column C: enter first y-value, using a formula, then select cell, hover over lower right corner, double-click
- ▶ Graph: put cursor in table, Ctrl-A, Insert, Chart



## Improving the Excel Graph

- ▶ To control scales and max/min:
  - ▶ Right-click, choose Format Chart Area.
  - ▶ Chart Options, choose square, Size, enter height and width, lock aspect ratio.



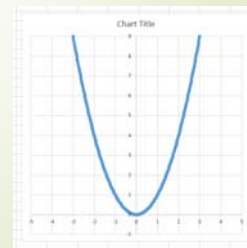
## Improving the Excel Graph

- ▶ Chart Options drop-down menu, choose an Axis.
- ▶ Bars, Axis Options, enter bounds (not Auto).
- ▶ Repeat for the other axis.

Two axes

Choose bars

Now we have a squared-up graph



## Parallel Curve Math

- ▶ Get slope of tangent line (or close secant line), then slope of normal

$$m_{\text{tan}} = \frac{y_2 - y_1}{x_2 - x_1} \quad m_{\text{normal}} = \frac{-1}{m_{\text{tan}}}$$

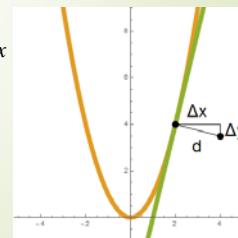
- ▶ Given distance  $d$ , decompose into  $dx$  and  $dy$

$$d^2 = (\Delta x)^2 + (\Delta y)^2 \quad \text{and} \quad m_{\text{normal}} = \frac{\Delta y}{\Delta x}$$

$$\Rightarrow \Delta x = \frac{d}{\sqrt{1 + m_{\text{normal}}^2}} \quad \text{and} \quad \Delta y = m_{\text{normal}} \Delta x$$

- ▶ Describe new point

$$x_{\text{new}} = x + \Delta x \quad \text{and} \quad y_{\text{new}} = y + \Delta y$$



## Parallel Curve Formulas in Excel

- Slope:
- Normal slope:
- Delta-x (for d=3):
- Delta-y:
- New x:
- New y:
- Copy all columns down.

x	y	m-tan	m-norm	delta-x	delta-y	new-x	new-y
-3.2	10.24						
-3.18	10.1124	-6.36	0.157233	2.963591	0.465973	-0.21641	10.57837
-3.16	9.9856	-6.32	0.158228	2.963137	0.468851	-0.19686	10.45445

## Parallel Curve Graph in Excel

- Select data
- Note original series values, then select Add
- Enter new values (one less row on each end)
- If needed, adjust axes values of graph

Chart data range: =Sheet1!\$B\$3:\$C\$32

Legend Entries (Series)

- Series1

Edit Series

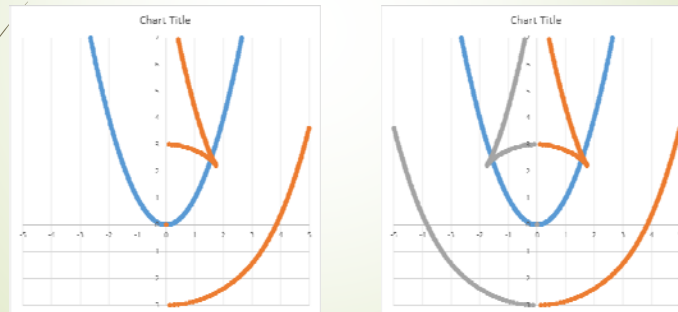
Series name: = "Parallel"

Series X values: =Sheet1!\$H\$4:\$H\$32

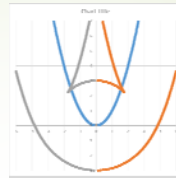
Series Y values: =Sheet1!\$I\$4:\$I\$32

## First Attempt at a Graph

- SURPRISE: Not quite what we expected.
- Need 2 new columns:  $x_{new} = x - \Delta x$  and  $y_{new} = y - \Delta y$



## Is this a polynomial function?



- No, it does not pass the vertical line test.
- Even if it did, polynomials do not have cusps.
- But yes (SURPRISE), sort of...
- If you think of the curve as  $f(x, y) = 0$ , that is, as
- A level curve of a bivariate polynomial function.

## Parametric Form

- Beginning with  $y = f(x)$ , which is  $(t, f(t))$
- Recall  $\Delta x = \frac{d}{\sqrt{1+m_{\text{normal}}^2}}$  and  $\Delta y = m_{\text{normal}} \Delta x$
- But  $m_{\text{normal}} = \frac{-1}{f'(x)}$
- So  $(x + \Delta x, y + \Delta y) = \left( t + \frac{d f'(t)}{\sqrt{1+(f'(t))^2}}, f(t) - \frac{d}{\sqrt{1+(f'(t))^2}} \right)$

## Parametric Polynomial System

- Rewrite new  $(x, y) = \left( t + \frac{d f'(t)}{\sqrt{1+(f'(t))^2}}, f(t) - \frac{d}{\sqrt{1+(f'(t))^2}} \right)$
- As a system  $x = t + \frac{d f'(t)}{\sqrt{1+(f'(t))^2}}$  and  $y = f(t) - \frac{d}{\sqrt{1+(f'(t))^2}}$
- Isolate radical terms and square both sides:
 
$$(x-t)\sqrt{1+(f'(t))^2} = d f'(t) \quad \text{and} \quad (y-f(t))\sqrt{1+(f'(t))^2} = -d$$

$$(x-t)^2(1+(f'(t))^2) = d^2(f'(t))^2 \quad \text{and} \quad (y-f(t))^2(1+(f'(t))^2) = d^2$$

$$\begin{cases} (x-t)^2(1+(f'(t))^2) - d^2(f'(t))^2 = 0 \\ (y-f(t))^2(1+(f'(t))^2) - d^2 = 0 \end{cases}$$

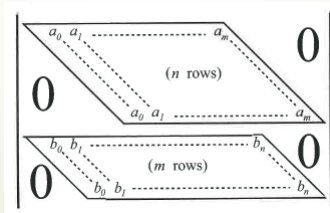


## Bivariate Polynomial Form

- Any parametric polynomial system

$$\begin{cases} a_0 t^m + a_1 t^{m-1} + \dots + a_m = 0 \\ b_0 t^n + b_1 t^{n-1} + \dots + b_n = 0 \end{cases}$$

- can be transformed into a bivariate polynomial form by using the resultant, which is the determinant:



## Our Parallel Curve, Parametrically

$$f(x) = x^2, \text{ with } d = 3$$

- Original curve, parametric:  $(t, t^2)$
- Slope is  $f'(x) = 2x$
- Parallel curve, parametric:  $\left( t + \frac{3(2t)}{\sqrt{1+(2t)^2}}, t^2 - \frac{3}{\sqrt{1+(2t)^2}} \right)$
- System:  $x = t + \frac{6t}{\sqrt{1+4t^2}}$  and  $y = t^2 - \frac{3}{\sqrt{1+4t^2}}$

$$(x-t)^2(1+4t^2) - 36t^2 = 0 \quad \text{and} \quad (y-t^2)^2(1+4t^2) - 9 = 0$$

$$\begin{cases} 4t^4 - 8xt^3 + (4x^2 - 35)t^2 - 2xt + x^2 = 0 \\ 4t^6 + (1-8y)t^4 + (4y^2 - 2y)t^2 + (y^2 - 9) = 0 \end{cases}$$

### Our Parallel Curve's Resultant

System: 
$$\begin{cases} 4t^4 - 8xt^3 + (4x^2 - 35)t^2 - 2xt + x^2 = 0 \\ 4t^6 + (1 - 8y)t^4 + (4y^2 - 2y)t^2 + (y^2 - 9) = 0 \end{cases}$$

Resultant:

$$\det \begin{bmatrix} 4 & -8x & (4x^2-35) & -2x & x^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -8x & (4x^2-35) & -2x & x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -8x & (4x^2-35) & -2x & x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -8x & (4x^2-35) & -2x & x^2 & 0 & 0 \\ 4 & 0 & (1-8y) & 0 & (4y^2-2y) & 0 & (y^2-9) & 0 & 0 & 0 \\ 0 & 4 & 0 & (1-8y) & 0 & (4y^2-2y) & 0 & (y^2-9) & 0 & 0 \\ 0 & 0 & 4 & 0 & (1-8y) & 0 & (4y^2-2y) & 0 & (y^2-9) & 0 \\ 0 & 0 & 0 & 4 & 0 & (1-8y) & 0 & (4y^2-2y) & 0 & (y^2-9) \end{bmatrix}$$

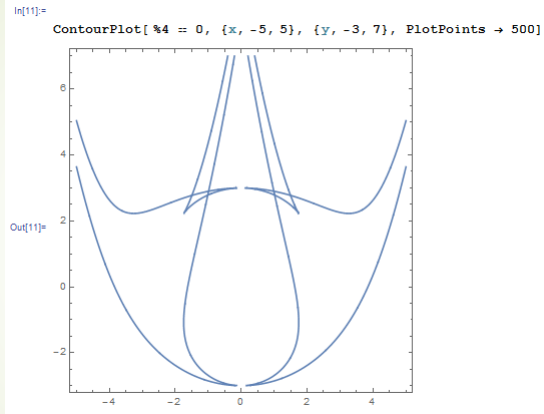
### Our Parallel Curve's Bivariate Polynomial

With the help of Mathematica

```
In[4]:= Resultant[(x - t)^2 (1 + 4 t^2) - 36 t^2, (y - t^2)^2 (1 + 4 t^2) - 9, t]
Out[4]= 2 504 040 397 056 - 1 958 472 566 784 x^2 + 558 274 125 312 x^4 - 82 011 543 552 x^6 + 6 575 323 392 x^8 - 283 336 704 x^10 + 5 308 416 x^12 - 1 148 454 236 160 y + 488 761 537 536 x^2 y - 45 575 073 792 x^4 y - 1 377 285 120 x^6 y + 367 939 584 x^8 y - 15 925 248 x^10 y - 359 145 570 816 y^2 + 316 760 647 680 x^2 y^2 - 78 921 423 360 x^4 y^2 + 8 802 680 832 x^6 y^2 - 467 804 160 x^8 y^2 + 10 616 832 x^10 y^2 + 240 162 693 120 y^3 - 69 427 860 480 x^2 y^3 + 828 776 448 x^4 y^3 + 634 355 712 x^6 y^3 - 37 158 912 x^8 y^3 - 12 502 128 384 y^4 - 12 641 329 152 x^2 y^4 + 2 512 207 872 x^4 y^4 - 148 635 648 x^6 y^4 + 5 308 416 x^8 y^4 - 10 834 145 280 y^5 + 1 871 216 640 x^2 y^5 + 180 486 144 x^4 y^5 - 21 233 664 x^6 y^5 + 2 340 347 904 y^6 + 180 486 144 x^2 y^6 + 31 850 496 x^4 y^6 - 185 794 560 y^7 - 21 233 664 x^2 y^7 + 5 308 416 y^8
```

- A 12<sup>th</sup> degree polynomial
- 39 terms

## Graph of the Resultant



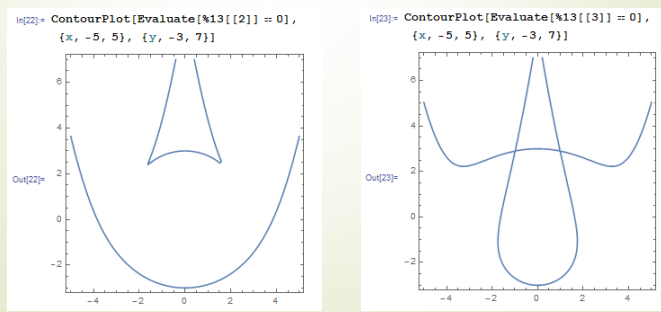
- Another SURPRISE: ... 4 branches, not 2
- We squared both sides, extraneous solutions

## Factoring the Resultant

- Factoring

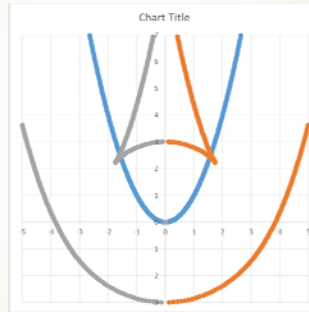
```
In[19]= Factor[%4]
Out[19]= 20736 (-12321 + 3708 x^2 - 431 x^4 + 16 x^6 + 2664 y +
70 x^2 y - 40 x^4 y + 1225 y^2 - 256 x^2 y^2 + 16 x^4 y^2 - 296 y^3 - 32 x^2 y^3 + 16 y^4)
(-9801 + 4716 x^2 - 423 x^4 + 16 x^6 + 2376 y - 234 x^2 y - 8 x^4 y + 945 y^2 -
320 x^2 y^2 + 16 x^4 y^2 - 264 y^3 - 32 x^2 y^3 + 16 y^4)
```

- Graphing Each Factor (6<sup>th</sup> degree, 13 terms)



## The Answer Page

- The Curve Parallel to  $y = x^2$ , at a distance of 3, is:



$$16x^6 + 16x^4y^2 - 40x^4y - 32x^2y^3 - 431x^4 - 256x^2y^2 + 16y^4 + 70x^2y - 296y^3 + 3708x^2 + 1225y^2 + 2664y - 12321 = 0$$