AND YOU THOUGHT THE BIRTHDAY PROBLEM WAS ONLY A CURIOSITY?

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THE BIRTHDAY PROBLEM

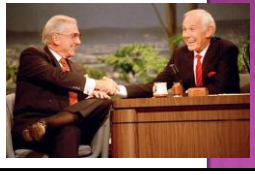
How many people must be in a room before the probability of at least two sharing a birthday is greater than 50%?

JOHNNY CARSON

Johnny Carson's stab at it …

http://www.cornell.edu/video/the-tonight-show-with-johnny-carson-feb-6-1980-excerpt

Also Feb 7, Feb 8



Johnny Carson, 1925-2005
Ed McMahon, 1923-2009

SOLUTION

- P(at least 2 share) = 1 – P(no one shares)
- To find P(no one shares), do probabilities of choosing a non-matching birthday:

$$= \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \left(1 - \frac{n-1}{365}\right)$$

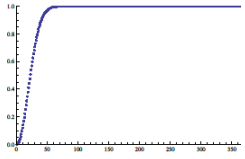
$$= \frac{365 \times 364 \times 363 \times \cdots \times (365-(n-1))}{365^n}$$

$$= \frac{365!}{(365-n)! \times 365^n} = \frac{{}_{365}P_n}{365^n}$$

- So P(at least 2 share) $= 1 - \frac{{}_{365}P_n}{365^n}$

STANDARD RESULT

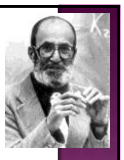
- P(at least 2 share) $= 1 - \frac{{}_{365}P_n}{365^n}$



n	P(sharing)
5	2.71%
10	11.69%
15	25.29%
20	41.14%
25	56.87%
30	70.63%
35	81.44%
40	89.12%
45	94.10%
50	97.04%

- For n = 23, P(at least 2 share) = 50.73%

REACTIONS?



The birthday problem used to be a splendid illustration of the advantages of pure thought over mechanical manipulation …

… what calculators do not yield is understanding, or mathematical facility, or a solid basis for more advanced, generalized theories.

--- Paul Halmos (1916-2006), *I Want to Be a Mathematician*, 1985

MY QUESTIONS? (MY OUTLINE)

- Is this only a curiosity?
- How was this computed historically?
- What is the underlying distribution?
- Generalizations?
- Applications?

NOT THEORETICAL?

[After solving the Birthday Problem]

"The next example in this section not only possesses the virtue of giving rise to a somewhat surprising answer, but it is also of theoretical interest."

- Sheldon Ross, *A First Course in Probability*, 1976

NOVEL? SO-CALLED?

[After developing a formula for the probability that no point appears twice when sampling with replacement]

"A novel and rather surprising application of [the formula] is the so-called birthday problem."

- Hoel, Port, & Stone, *Introduction to Probability Theory*, 1971

THE LEGENDARY SOURCE

Richard von Mises (1883-1953) often gets credit for posing it in 1939, but …

… he sought the expected number of repetitions as a function of the number of people.

BUT EVEN EARLIER …

Ball & Coxeter included it in their 11th edition of *Mathematical Recreations and Essays*, published in 1939.

They gave credit to Harold Davenport (1907-1969), who shared it about 1927.

But he did not think he was the originator either.

COMPUTING BEFORE CALCULATORS?

So how did they do the computations back then?

Paul Halmos: "The birthday problem used to be a splendid illustration of the advantages of pure thought over mechanical manipulation …"

A SERIES APPROXIMATION

- P(no one shares)

$$= \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{365-(n-1)}{365}$$

$$= \left(1-\frac{0}{365}\right) \times \left(1-\frac{1}{365}\right) \times \left(1-\frac{2}{365}\right) \times \cdots \times \left(1-\frac{n-1}{365}\right)$$

- Recall the Taylor Series: $e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots$
- First-order approximation: P(no one shares)

$$= e^{-0/365} \times e^{-1/365} \times e^{-2/365} \times \cdots \times e^{-(n-1)/365}$$

$$= 1 \times e^{-(1+2+\cdots+(n-1))/365}$$

$$= e^{-(n(n-1)/2)/365}$$

AN APPROXIMATE SOLUTION

Solving P(at least 2 share) > 0.50

$$1 - e^{-(n(n-1)/2)/365} > 0.5$$

$$e^{-(n(n-1)/2)/365} < 0.5$$

$$\frac{-n(n-1)}{2(365)} < \ln(0.5)$$

$$\boxed{n(n-1) > (365)\ln 4} \approx 505.997$$

$$n^2 - n - 506 > 0$$

$$n > \frac{1+\sqrt{1+4(506)}}{2} = \frac{1+\sqrt{2025}}{2} = 23$$

OR MORE GENERALLY…

$$n(n-1) > d \ln 4$$

$$n^2 - n - d\ln 4 > 0$$

$$n > \frac{1+\sqrt{1+4d\ln 4}}{2}$$

$$n > 0.5 + \sqrt{0.25 + d\ln 4}$$

$$n \approx 0.5 + 1.177\sqrt{d}$$

… OR WITH MENTAL MATH

$$n > 0.5 + \sqrt{0.25 + d \ln 4}$$

$$n \approx 0.5 + 1.177\sqrt{d}$$

$$n \approx 1.2\sqrt{d}$$

So *n* is roughly proportional to the square root of *d*, with proportionality constant about 1.2.

$$n \approx 1.2\sqrt{365} \approx 22.93$$

$$n \approx 0.5 + 1.177\sqrt{365} \approx 22.99$$

(Pat'sBlog attributes the factor 1.2 to Persi Diaconis)

A HEURISTIC EXPLANATION

- P(2 people not sharing) = $\frac{364}{365}$
- Among n people, number of pairs = ${}_nC_2 = \frac{n(n-1)}{2}$
  - For 23 people, there are 253 pairs!
  - That's 253 chances of a birthday match!
- P(2 of n people sharing) ≈ $1 - \left(\frac{364}{365}\right)^{\frac{n(n-1)}{2}}$
  - Alarm Bells! Independence assumed!

IGNORING THE ALARM

- P(2 of n people sharing) ≈ $1 - \left(\frac{364}{365}\right)^{\frac{n(n-1)}{2}}$
  - Alarm Bells temporarily suppressed!
- Solving: $1 - \left(\frac{364}{365}\right)^{\frac{n(n-1)}{2}} > 0.5$

$$n^2 - n - 505.3 > 0$$

- gives n > 22.98

UNDERLYING DISTRIBUTION?

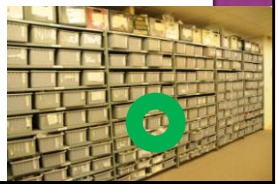
- Discrete or Continuous?

Discrete Distribution	Random Variable	Assumptions
Binomial	No. of Successes	Fixed n, constant p, independent
Hypergeometric	No. of Successes	Fixed n, p varies, dependent
Poisson	No. of Successes	Fixed time, constant λ, independent
Geometric	First Success	n varies, constant p, independent
Negative Binomial	k-th Success	n varies, constant p, independent
Multinomial	No. of Each Type	Fixed n, constant p, independent

- These answer the wrong question

BALLS IN BINS

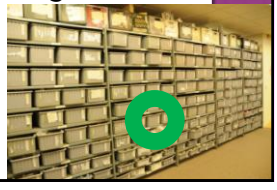
- Place *n* indistinguishable balls into *d* distinguishable bins.
- Balls → People, Bins → Birthdays
- When does one bin contain two (or more) balls?



- Bernoulli ???
- Multinomial ???

FIRST MATCH

- Want "first match" !
- AKA: occupancy problem, collision counting



THE FIRST MATCH PATTERN

- How many trials to get the first match?
- Variables:
  - x = number of trials to get the first match
  - d = number of equally likely options (birthdays)
- Pattern:

Not Not Not ... Not Match

$$\frac{d}{d}\times\frac{d-1}{d}\times\frac{d-2}{d}\times\cdots\times\frac{d-(x-2)}{d}\times\frac{x-1}{d}$$

THE FIRST MATCH PROBABILITY

- Pattern:

Not Not Not ... Not Match

$$\frac{d}{d}\times\frac{d-1}{d}\times\frac{d-2}{d}\times\cdots\times\frac{d-(x-2)}{d}\times\frac{x-1}{d}$$

- Probability:

$$P(X=x)=\frac{(x-1)}{d^{x}}\left({}_{d}P_{x-1}\right) \quad \text{for} \quad x\in\{2,3,4,\ldots,d+1\}$$

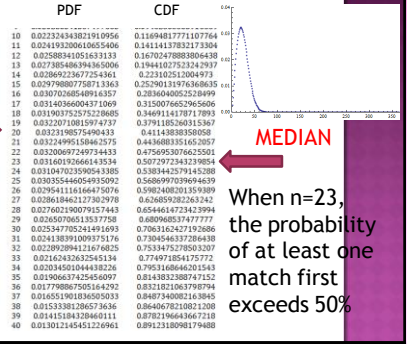
PROBABILITY DISTRIBUTIONS

MEAN

E(X) ≈ 24.6166

MODE

The probability of obtaining the first match is highest when n = 20



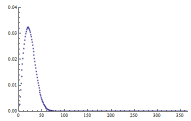
MEDIAN

When n=23, the probability of at least one match first exceeds 50%

THE MODE ANALYTICALLY

- The PDF is discrete, so avoid calculus.
- For which x is P(x) > P(x+1) ?

$$\frac{x-1}{d^x}({}_dP_{x-1}) > \frac{x}{d^{x+1}}({}_dP_x)$$
$$\frac{(x-1)}{d^x}\frac{d!}{(d-x+1)!} > \frac{x}{d^{x+1}}\frac{d!}{(d-x)!}$$
$$d(x-1) > x(d-x+1)$$
$$x^2 - x - d > 0$$
$$x > \frac{1+\sqrt{1+4d}}{2}$$



EXACT FORMULA!

- For d=365, we get x>19.61, so 19 or 20
- Approximating: $x > \sqrt{d}$ for large *d*

THE MEAN (EXPECTED VALUE)

- Since then $P(X=x) = \frac{(x-1)}{d^x}({}_dP_{x-1})$ for $x \in \{2,3,4,\ldots,d+1\}$

$$E(X) = \sum_{x=2}^{d+1} x\frac{x-1}{d^x}({}_dP_{x-1})$$

- which is related to Ramanujan's Q-function, specifically $E(X) = 1 + Q(d)$
- which is asymptotic:

$$E(X) \sim 1 + \sqrt{\frac{\pi d}{2}} - \frac{1}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2d}} - \frac{4}{135d} + \cdots$$

- and for d=365, we get:

$$E(X) \approx \sqrt{\frac{365\pi}{2}} + \frac{2}{3} + \frac{1}{12}\sqrt{\frac{\pi}{730}} - \frac{4}{49275} + \cdots \approx 24.6166$$

BIRTHDAY PROBLEM ON JUPITER?

- A Jovian day = 9.9259 Earth-hours
- A Jovian year = 11.86231 Earth-years

$$1\,Jyr = 11.86231\,Eyrs\left(\frac{365.24\,Edays}{1\,Eyr}\right)\left(\frac{24\,Ehrs}{1\,Eday}\right)\left(\frac{1\,Jday}{9.9259\,Ehrs}\right)$$
$$\approx 10476\,Jdays$$
$$P(X=x) = \frac{(x-1)}{10476^x}({}_{10476}P_{x-1}) \quad \text{for} \quad x \in \{2,3,4,\ldots,10477\}$$

- Both 10476^{25} and ${}_{10476}P_{25}$ cause a TI-84 overflow!

JOVIAN MEDIAN ESTIMATES

- Proportionality?

$$n \approx 1.2\sqrt{10476} \approx 122.823$$

- Heuristically? $1-\left(\frac{10475}{10476}\right)^{\frac{n(n-1)}{2}} > 0.5$

$$n(n-1) > \frac{\ln 4}{\ln\left(\frac{10476}{10475}\right)}$$

$$n \approx 121.009$$

MORE JOVIAN MEDIAN ESTIMATES

- Series Approximation:

$$\frac{10476}{10476} \times \frac{10475}{10476} \times \cdots \times \frac{10476-(n-1)}{10476} < 0.5$$

$$n(n-1) > (10476)\ln 4$$

$$n \approx 121.012$$

- Mathematica: for n = 121, P(match) = 50.13%

JOVIAN BIRTHDAY AVERAGES

- Mean:

$$E(X) \approx \sqrt{\frac{\pi d}{2}} + \frac{2}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2d}} - \frac{4}{135d} + \cdots \approx 128.947$$

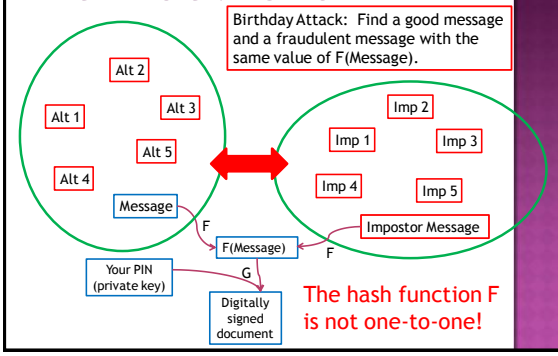
- Median:

$$n \approx \frac{1+\sqrt{1+4d\ln 4}}{2} \approx 121.012$$

- Mode:

$$x \approx \frac{1+\sqrt{1+4d}}{2} \approx 102.854$$

DIGITAL SIGNATURES



MATCHING BETWEEN 2 SETS

Given *n* good messages, *n* fraudulent messages, and a hash function with *d* possible outputs, what is the probability of a match between 2 sets?

- Assume $n \ll d$, so *n* outputs are probably all different.
- P(message 1 in set 1 doesn't match set 2) = $1-\frac{n}{d}$

MATCHING BETWEEN 2 SETS

- P(no message in set 1 matches anything in set 2) = $\left(1-\frac{n}{d}\right)^n \approx \left(e^{-n/d}\right)^n = e^{-n^2/d}$
- 50% probability of at least one match between the sets: $1-e^{-n^2/d} > 0.50$

$$n^2 > d \ln 2$$

$$n \approx 0.83\sqrt{d}$$

- Probability *p* of at least one match between the sets: $n \approx \sqrt{d \ln\left(\frac{1}{1-p}\right)}$

ATTACKING A 16-BIT HASH

- With a 16-bit hash function, there are $2^{16} = 65,536$ possible outputs
- 50% probability of a match with messages: $n \approx 0.83\sqrt{65536} \approx 213$
- 1% probability of a match with messages $n \approx \sqrt{65536 \ln\left(\frac{1}{0.99}\right)} \approx 26$

IMPROVING THE DEFENSE

Increase the size of the hash function:

- With 64-bits, $2^{64} = 18,446,744,073,709,551,616$ outputs are possible
- 50% probability of a match with 3.5 billion messages (11 messages per US citizen)
- 1% probability of a match with 431 million messages

ANSWERS TO MY QUESTIONS

- Is this only a curiosity?
  - No, it's part of a class of problems
- How was this computed historically?
  - Using approximations (including Taylor Series)
- What is the underlying distribution?
  - "First Match" Distribution
- Generalizations?
  - Vary days in a year (other planets)
- Applications?
  - Digital signatures

THANK YOU!

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- A PDF of the presentation is available at:
  http://www.milefoot.com/about/presentations/birthday.pdf