THIRTY-FOUR FLAVORS OF
CUBIC PLANE CURVES

Steven J. Wilson, Instructor
Johnson County Community College

A cubic plane curve is a graph of an equation of the form:

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0$$

<u>Some TI-81 Hints</u>:

Two programs are given below, each of which sets the RANGE variables of the TI-81. These programs enable the coordinates of most points to be given to one decimal place, making the TRACE feature easier to use. (Courtesy of Nancy H. Olson)

```
Prgm1:RANGE
:Disp "FACTOR"
:Input F
:-4.8F→Xmin
:4.7F→Xmax
:F→Xscl
:-3.2F→Ymin
:3.1F→Ymax
:F→Yscl
:End
```

```
Prgm2:AUTORNGE
:-4.8→Xmin
:4.7→Xmax
:1→Xscl
:-3.2→Ymin
:3.1→Ymax
:1→Yscl
:End
```

When graphing "functions" of the form $f(x) \pm \sqrt{g(x)}$, break the problem into parts as shown below. Then turn off the display of the graphs of Y_1 and Y_2, and you will graph only Y_3 and Y_4.

$Y_1=f(x)$
$Y_2=g(x)$
$Y_3=Y_1+\sqrt{}Y_2$
$Y_4=Y_1-\sqrt{}Y_2$

Thirty-Four Flavors of Cubic Plane Curves

Steven J. Wilson

1. $x^3 - 3x - y = 0$

2. $x^2y = 1$
3. $x^2y - 4y = 4$
4. $x^2y - 2x - 4y = 0$
5. $x^3 - xy = 1$

6. $x^3 - y^2 = -2$
7. $x^3 - y^2 - 3x = -2$
8. $x^3 - y^2 = 0$
9. $x^3 - y^2 - 3x = 0$
10. $x^3 - y^2 - 3x = 2$

11. $3x^3 + 5xy^2 + 14x^2 - x = -8$
12. $3x^3 + 5xy^2 + 14x^2 + 19x = -8$
13. $x^3 + xy^2 + 3x^2 + 3x = -1$
14. $3x^3 + 5xy^2 - 4x^2 - 10y^2 - 45x = -90$
15. $9x^3 + 15xy^2 - 12x^2 - 30y^2 - 128x = -256$

16. $xy^2 + x - 6y = 0$
17. $x^3 + xy^2 - 5x^2 + 3x - 4y = -5$
18. $x^3 + xy^2 - 2x^2 - 2y = -2$
19. $x^3 + xy^2 - 5x^2 - 8y = -20$
20. $x^3 + xy^2 - 4x^2 + 3x - 4y = -4$

21. $3x^3 - 5xy^2 - 4x^2 + 10y^2 - 21x = -42$
22. $3x^3 - 5xy^2 - 4x^2 + 10y^2 - 45x = -90$
23. $9x^3 - 15xy^2 - 12x^2 + 30y^2 - 128x = -256$
24. $3x^3 - 5xy^2 + 14x^2 + 20x = -8$
25. $x^3 - xy^2 + 3x^2 + 3x = -1$
26. $6x^3 - 10xy^2 + 28x^2 + 39x = -16$
27. $3x^3 - 5xy^2 + 14x^2 + 19x = -8$

28. $xy^2 - x^2 = 4$
29. $xy^2 - x^2 = -4$
30. $xy^2 - x^2 + 4x = 4$
31. $xy^2 - x^2 - 6x - 4y = 9$
32. $xy^2 - x^2 - 3x - 2y = 3$
33. $xy^2 - x^2 - 5x = 4$
34. $xy^2 - x^2 - 4x = 4$

Questions to ponder:

1. What types of singularities can occur when an equation of the form $y = f(x)$ is transformed into the equation $y^2 = f(x)$? Can you classify the type of singularity only using the values of the derivatives of $f(x)$?

2. If you know the graph of the equation $f(x,y) = 0$, can you predict the graph of the equation $f(x,y^2) = 0$?

3. Can any equation of a cubic plane curve be transformed, by a suitable rotation of the coordinate axes, into an equation having no y^3 term? If so, can you give an explicit formula for the angle of rotation?

4. Given a collection of graphs of cubic plane curves, can you classify them, according to their main features, into a finite number of types? You must decide what features qualify as main features of the graphs. Some possibilities might include cusps, nodes, isolated points, number of branches, inflection points, and asymptotes.

5. Given an equation of a cubic plane curve, can you predict its type of graph from the coefficients of the equation only?

6. Many equations of cubic plane curves, when solved for y, result in expressions for which y is double-valued. Consider the collection of functions obtained when each double-valued expression is broken into two single-valued functions. What types of unusual behavior can these functions exhibit?

7. Many special plane curves are cubics. Find out about the properties of these cubic curves: Cissoid of Diocles, Folium of Descartes, L'Hospital's Cubic, Newton's Trident, Right Strophoid, Serpentine, Trisectrix of Catalan, Trisectrix of Maclaurin, Tschirnhausen's Cubic, and the Witch of Agnesi.

Thirty-Four Flavors of Cubic Plane Curves

Steven J. Wilson