

## Confessions of an Integermaniac

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## A Bit of Perspective

- Small minds discuss persons.
- Average minds discuss events.
- Great minds discuss ideas.
- Really great minds discuss mathematics.

From AMATYC News, January 2009, attributed to an unknown source.

## The Four Fours Problem

Create the integers from 1 to 100 using four fours and any operations.

- $1 = 4 / 4 + 4 - 4$
- $2 = 4 / 4 + 4 / 4$
- $3 = (4 + 4 + 4) / 4$
- $4 = 4 + 4 \times (4 - 4)$

## Four Fours ...

- is an example of recreational mathematics
- is easy and fun for beginners
- can be challenging for advanced students
- can become addictive
- solutions can be found online in many places

## A Long History

David Singmaster, Chronology of Recreational Mathematics:

- 1743, Dilworth: **The Schoolmaster's Assistant** – first Four Fours type problem.
- 1881, General Four Fours problem appears in *Knowledge* – previously only specific cases had been set.

## The Web Site

<http://www.milefoot.com/math/integermania>

$\frac{1}{4+4}$ <p>1 (1.0) Carolyn Neptune, 306 Prairie Village, KS</p>	$\frac{2}{4 \times 4}$ <p>2 (1.0) Carolyn Neptune, 406 Prairie Village, KS</p>	$\frac{3}{4 \times 4 - 4}$ <p>3 (1.0) Dave Jones, 506 Coventry, England</p>	$\frac{4}{(4-4) \times 4 + 4}$ <p>4 (1.0) Dave Jones, 506 Coventry, England</p>
$\frac{11}{\frac{4}{4} + \frac{4}{4}}$ <p>11 (2.2) Matt Watters, 406 Prairie Village, KS</p>	$\frac{12}{\left(\frac{4}{4} - \frac{4}{4}\right) \times 4}$ <p>12 (1.0) Levi Sak, 606 San Antonio, TX</p>	$\frac{13}{\frac{4}{44} + 4}$ <p>13 (2.4) Steve Wilson, 706 Raytown, MO</p>	$\frac{14}{\frac{4+4}{4} - 4}$ <p>14 (2.4) Steve Wilson, 706 Raytown, MO</p>
$\frac{21}{4 \times 4\% - 4}$ <p>21 (2.2) Dave Jones, 1006 Coventry, England</p>	$\frac{22}{\frac{4}{4} + 4}$ <p>22 (2.4) Steve Wilson, 706 Raytown, MO</p>	$\frac{23}{41 - 4^4 - 4}$ <p>23 (3.2) Pellegri Paolo, 708 Matina Franca, Italy</p>	$\frac{24}{4 \times 4 + 4 + 4}$ <p>24 (1.0) Carolyn Neptune, 406 Prairie Village, KS</p>
$\frac{31}{\sqrt[4]{4} - \frac{4}{4}}$ <p>31 (3.2) Pellegri Paolo, 708 Matina Franca, Italy</p>	$\frac{32}{4 \times 4 + 4 \times 4}$ <p>32 (1.0) Levi Sak, 606 San Antonio, TX</p>	$\frac{33}{\sqrt[4]{4} + \frac{4}{4}}$ <p>33 (3.2) Pellegri Paolo, 708 Matina Franca, Italy</p>	$\frac{34}{44 - \frac{4}{4}}$ <p>34 (2.2) Dave Jones, 606 Coventry, England</p>

### Selecting a Best Solution

- Q: Since we can't post every possible solution, how do we define the best solution?
- A: The best solution will be the simplest in terms of the operations used (since the digits used are always identical).

### Exquisiteness

- The highest level operation used
- Level 1:  $4 + 4$ ,  $4 - 4$ ,  $4 \times 4$ ,  $4 / 4$
  - Level 2:  $44$ ,  $4.4$ ,  $4.\bar{4}$ ,  $4\%$
  - Level 3:  $4!$ ,  $\sqrt{4}$ ,  $\sqrt[4]{4}$ ,  $4\%$
  - Level 4:  $\log(4/4)$ ,  $\cot(\arctan(4\%))$
  - Level 5:  ${}_4!C_4$ ,  $\Gamma(4)$
- with a 0.2 surcharge for each unary operation

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$\frac{1(1.0)}{4+4}$ Carolyn Negtma, 306 Prairie Village, KS	$\frac{2(1.0)}{4 \times 4}$ Carolyn Negtma, 406 Prairie Village, KS	$\frac{3(1.0)}{4 \times 4 - 4}$ Dave Jones, 506 Coveasy, England	$\frac{4(1.0)}{(4-4) \times 4 + 4}$ Dave Jones, 506 Coveasy, England
$\frac{11(2.2)}{4 + \frac{4}{4}}$ Matt Winters, 406 Prairie Village, KS	$\frac{12(1.0)}{\left(\frac{4}{4-4}\right) \times 4}$ Levi Saf, 606 San Antonio, TX	$\frac{13(2.4)}{\frac{4}{.44} + 4}$ Steve Wilson, 706 Raytown, MO	$\frac{14(2.4)}{\frac{4+4}{4} - 4}$ Steve Wilson, 706 Raytown, MO
$\frac{15(2.2)}{4 \times 4\%}$ Dave Jones, 1006 Coveasy, England	$\frac{17(2.4)}{\frac{4+4}{4} + 4}$ Steve Wilson, 706 Raytown, MO	$\frac{19(3.2)}{4! - 4^{4-4}}$ Pellegrini Paolo, 708 Martina Franca, Italy	$\frac{24(1.0)}{4 \times 4 + 4 + 4}$ Carolyn Negtma, 406 Prairie Village, KS
$\frac{31(3.2)}{\sqrt[4]{4} - 4}$ Pellegrini Paolo, 708 Martina Franca, Italy	$\frac{32(1.0)}{4 \times 4 + 4 \times 4}$ Levi Saf, 606 San Antonio, TX	$\frac{33(3.2)}{\sqrt[4]{4} + \frac{4}{4}}$ Pellegrini Paolo, 708 Martina Franca, Italy	$\frac{34(2.2)}{44 - 4}$ Dave Jones, 606 Coveasy, England

### How Many Solutions?

- For the set  $\{a, b, c, d\}$ , with operations  $\{+, -, \times, \div\}$ , there are at most 7680 level 1 solutions.
- In general, with  $n$  distinct values and  $k$  binary operations, there are at most  $\frac{k^{n-1}(2n-2)!}{(n-1)!}$  solutions.

### Proof of Formula $\frac{k^{n-1}(2n-2)!}{(n-1)!}$

Product of the number of ways to ...

- arrange the  $n$  values  $n!$
- choose the  $(n-1)$  operations  $k^{n-1}$
- insert parentheses  $\frac{1}{n} \binom{2n-2}{n-1}$

### Proof, Part 2

- Let  $P_{n-1}$  be the number of parenthesizings of  $m_1 p_1 m_2 \dots m_{n-1} p_{n-1} m_n$
- $(m_1 p_1 m_2 \dots p_{q-1} m_q) p_q (m_{q+1} p_{q+1} m_{q+2} \dots p_{n-1} m_n)$
- Recursive relation:

$$P_{n-1} = \sum_{q=1}^{n-1} P_{q-1} P_{n-q-1}, P_0 = 1$$

$$P_{n+1} = \sum_{q=0}^n P_q P_{n-q}, P_0 = 1$$

### Proof, Part 3

- Consider the power series

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} P_n x^n = 1 + \sum_{n=0}^{\infty} P_{n+1} x^{n+1} \\ &= 1 + \sum_{n=0}^{\infty} \left( \sum_{q=0}^n P_q P_{n-q} \right) x^{n+1} \\ &= 1 + \sum_{q=0}^{\infty} \left( P_q x^{q+1} \sum_{n=q}^{\infty} P_{n-q} x^{n-q} \right) \\ &= 1 + x S(x) \sum_{q=0}^{\infty} P_q x^q = 1 + x (S(x))^2 \end{aligned}$$

### Proof, Part 4

- The solution of  $S(x) = 1 + x(S(x))^2$

- is  $S(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$

- whose power series expansion is

$$S(x) = \frac{1}{2x} \sum_{k=1}^{\infty} \frac{2(2k-2)!}{k!(k-1)!} x^k = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^k$$

- Thus  $P_n = \frac{1}{n+1} \binom{2n}{n}$  or  $P_{n-1} = \frac{1}{n} \binom{2n-2}{n-1}$

### Four Fours Solutions?

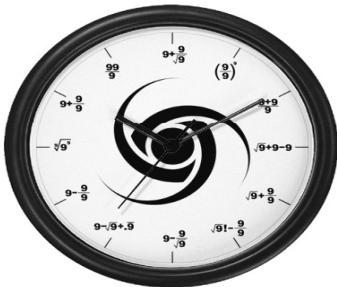
- Formula predicts at most 7680 at level 1
- Since digits are repeated, at most 320 at level 1
- Thanks to commutativity and coincidence, exactly 25 at level 1
- And at levels 2.0-2.8, in the thousands...

### Every Integer is Solvable!

Proof:

$$4 - 4 - \log \sqrt{\left(\frac{4}{4}\right) \% \dots \%} = -\log(10^{-2n})^{1/2} = n$$

### How Maniacal are You?



### How Maniacal are You?

- Apprentices: work sporadically or occasionally
- Journeymen: have worked consistently for at least six months
- Masters: work years, usually systematically, and may often use the computer

## How to Become a Master

- Know singles and pairs
- Record everything you find
- Don't just try for the next number
- Work all variations on a pattern
- Understand how operations work
- Use the computer

## How Operations Work

- Binary ops:  $3 + 5$ ,  $\sqrt[3]{8}$ ,  $4+7C_3$
- Unary ops:  $6\%$ ,  $4!$ ,  $\sqrt{4}$ ,  $\sinh(\ln 2)$
- Limited ops:

$$2.34 = 2,34, = 2 \frac{34}{100} = \frac{234}{100}$$

$$2.\overline{34} = 2,,34 = 2 \frac{34}{99} = \frac{232}{99}$$

$$2.\overline{3}4 = 2,3,4 = 2 \frac{3}{10} + \frac{4}{90} = \frac{211}{90}$$

## Hash Tables

- Create a table for each possible subset of digits:  $\{4\}$ ,  $\{4, 4\}$ ,  $\{4, 4, 4\}$ ,  $\{4, 4, 4, 4\}$
- Style: {integer, solution string, level}
- Include limited ops in each hash table
- Do unary ops on each hash table
- Do binary ops on the "smaller" hash tables to produce entries for the "larger" hash tables

## Hash Table Examples

- |                                              |                             |
|----------------------------------------------|-----------------------------|
| ■ $\{4, 4, 1.0\}$                            | ■ $\{44, 44, 2.0\}$         |
| ■ $\{.4, .4, 2.2\}$                          | ■ $\{4.4, 4.4, 2.0\}$       |
| ■ $\{4/9, .4\sim, 2.4\}$                     | ■ $\{0, 4 - 4, 1.0\}$       |
| ■ $\{2, \text{sqrt}(4), 3.2\}$               | ■ $\{1, 4 / 4, 1.0\}$       |
| ■ $\{2/3, \text{sqrt}(.4\sim), 3.6\}$        | ■ $\{8, 4 + 4, 1.0\}$       |
| ■ $\{24, 4!, 3.2\}$                          | ■ $\{16, 4 \times 4, 1.0\}$ |
| ■ $\{6, \Gamma(4), 5.2\}$                    | ■ $\{10, 4 / .4, 2.2\}$     |
| ■ $\{1, \Gamma(\text{sqrt}(4)), 5.4\}$       | ■ $\{100, 4 / (4\%), 2.2\}$ |
| ■ $\{120, \Gamma(\Gamma(4)), 5.4\}$          | ■ $\{9, 4! / 4\sim, 2.4\}$  |
| ■ $\{720, \Gamma(4)!, 5.4\}$                 |                             |
| ■ $\{0, \log(\Gamma(\text{sqrt}(4))), 5.6\}$ |                             |

## Care to Contribute?

20062002496 (3.0)  
6445  
In: Integer, 1987  
Solved, Reused

427497328 (3.2)  
4<sup>(14 + 4)</sup>  
In: Gen, 2009  
Solved, R2

gwp4(3.4)  
 $\left(\frac{4}{-4}\right)$  4470  
In: 30-Dec-06, 163  
Reused, R2

gwp4(3.5)  
 $\left(\frac{4}{-4}\right)$   $\sqrt[4]{2870}$   
In: 30-Dec-06, 209  
Created, 16

## Conference Special

- Get recognition for your work today!
- Submit your name, hometown, and some of your solutions before you leave, and they may be posted on the Integermania website.
- <http://www.milefoot.com/math/integermania>
- **Fine print:** Only one solution per integer will be posted. At most five solutions per person will be posted. The most exquisite solutions will be preferred. Only one submitter per solution will be posted. For this special offer, the webmaster reserves the right to decide which solutions will be posted.