Confessions of an Integermaniac

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Omaha, Nebraska

A Bit of Perspective

- Small minds discuss persons.
- Average minds discuss events.
- Great minds discuss ideas.
- Really great minds discuss mathematics.

From AMATYC News, January 2009, attributed to an unknown source.

The Four Fours Problem

Create the integers from 1 to 100 using four fours and any operations.

- 1 = 4 / 4 + 4 – 4
- 2 = 4 / 4 + 4 / 4
- 3 = (4 + 4 + 4) / 4
- 4 = 4 + 4 x (4 – 4)

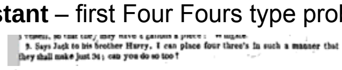
Four Fours …

- is an example of recreational mathematics
- is easy and fun for beginners
- can be challenging for advanced students
- can become addicting
- solutions can be found online in many places

A Long History

David Singmaster, Chronology of Recreational Mathematics:

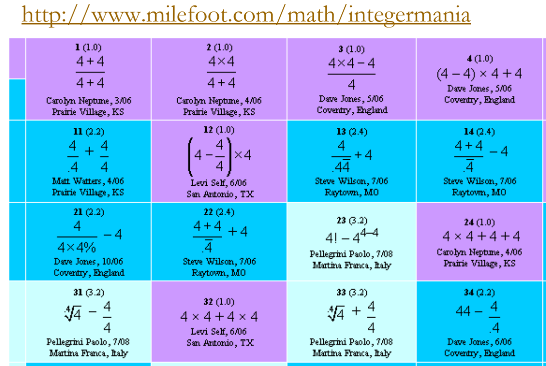
- 1743, Dilworth: **The Schoolmaster's Assistant** – first Four Fours type problem.


9. Says Jack to his brother Harry, I can place four three's in such a manner that they shall make just 34; can you do so too?

- 1881, General Four Fours problem appears in *Knowledge* – previously only specific cases had been set.

The Web Site

http://www.milefoot.com/math/integermania



1 (1.0) $\frac{4+4}{4+4}$ Carolyn Neptune, 3/06 Prairie Village, KS	**2** (1.0) $\frac{4\times4}{4+4}$ Carolyn Neptune, 4/06 Prairie Village, KS	**3** (1.0) $\frac{4\times4-4}{4}$ Dave Jones, 5/06 Coventry, England	**4** (1.0) $(4-4)\times4+4$ Dave Jones, 5/06 Coventry, England
11 (2.2) $\frac{4}{.4}+\frac{4}{4}$ Matt Watters, 4/06 Prairie Village, KS	**12** (1.0) $\left(4-\frac{4}{4}\right)\times4$ Levi Self, 6/06 San Antonio, TX	**13** (2.4) $\frac{4}{.4\bar{4}}+4$ Steve Wilson, 7/06 Raytown, MO	**14** (2.4) $\frac{4+4}{.\bar{4}}-4$ Steve Wilson, 7/06 Raytown, MO
21 (2.2) $\frac{4}{4\times4\%}-4$ Dave Jones, 10/06 Coventry, England	**22** (2.4) $\frac{4+4}{.\bar{4}}+4$ Steve Wilson, 7/06 Raytown, MO	**23** (3.2) $4!-4^{4-4}$ Pellegrini Paolo, 7/08 Martina Franca, Italy	**24** (1.0) $4\times4+4+4$ Carolyn Neptune, 4/06 Prairie Village, KS
31 (3.2) $\sqrt[.4]{4}-\frac{4}{4}$ Pellegrini Paolo, 7/08 Martina Franca, Italy	**32** (1.0) $4\times4+4\times4$ Levi Self, 6/06 San Antonio, TX	**33** (3.2) $\sqrt[.4]{4}+\frac{4}{4}$ Pellegrini Paolo, 7/08 Martina Franca, Italy	**34** (2.2) $44-\frac{4}{.4}$ Dave Jones, 6/06 Coventry, England

Selecting a Best Solution

- Q: Since we can't post every possible solution, how do we define the best solution?
- A: The best solution will be the simplest in terms of the operations used (since the digits used are always identical).

Exquisiteness

The highest level operation used

- Level 1: 4 + 4, 4 − 4, 4 x 4, 4 / 4
- Level 2: 44, 4.4, $4.\overline{4}$, 4%
- Level 3: 4!, $\sqrt{4}$, $\sqrt[.4]{4}$, 4‰
- Level 4: log(4/4), cot(arctan(4%))
- Level 5: ${}_{4!}C_4$, Γ(4)

with a 0.2 surcharge for each unary operation

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How Many Solutions?

- For the set { a, b, c, d }, with operations { +, −, x, ÷ }, there are at most 7680 level 1 solutions.
- In general, with n distinct values and k binary operations, there are at most $\frac{k^{n-1}(2n-2)!}{(n-1)!}$ solutions.

Proof of Formula $\frac{k^{n-1}(2n-2)!}{(n-1)!}$

Product of the number of ways to ...

- arrange the n values

$$n!$$

- choose the (n – 1) operations

$$k^{n-1}$$

- insert parentheses

$$\frac{1}{n}\binom{2n-2}{n-1}$$

Proof, Part 2

- Let P_{n-1} be the number of parenthesizings of $m_1p_1m_2 \ldots m_{n-1}p_{n-1}m_n$
- $(m_1p_1m_2 \ldots p_{q-1}m_q)p_q(m_{q+1}p_{q+1}m_{q+2} \ldots p_{n-1}m_n)$
- Recursive relation:

$$P_{n-1} = \sum_{q=1}^{n-1} P_{q-1}P_{n-q-1}\ , P_0 = 1$$

$$P_{n+1} = \sum_{q=0}^{n} P_q P_{n-q}\ , P_0 = 1$$

Proof, Part 3

- Consider the power series

$$S(x)=\sum_{n=0}^{\infty}P_n x^n = 1+\sum_{n=0}^{\infty}P_{n+1}x^{n+1}$$
$$=1+\sum_{n=0}^{\infty}\left(\sum_{q=0}^{n}P_q P_{n-q}\right)x^{n+1}$$
$$=1+\sum_{q=0}^{\infty}\left(P_q x^{q+1}\sum_{n=q}^{\infty}P_{n-q}x^{n-q}\right)$$
$$=1+x\,S(x)\sum_{q=0}^{\infty}P_q x^q = 1+x\left(S(x)\right)^2$$

Proof, Part 4

- The solution of $S(x)=1+x\left(S(x)\right)^2$
- is $S(x)=\dfrac{1-\sqrt{1-4x}}{2x}$
- whose power series expansion is

$$S(x)=\frac{1}{2x}\sum_{k=1}^{\infty}\frac{2(2k-2)!}{k!(k-1)!}x^k=\sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}x^k$$

- Thus $P_n=\dfrac{1}{n+1}\dbinom{2n}{n}$ or $P_{n-1}=\dfrac{1}{n}\dbinom{2n-2}{n-1}$

Four Fours Solutions?

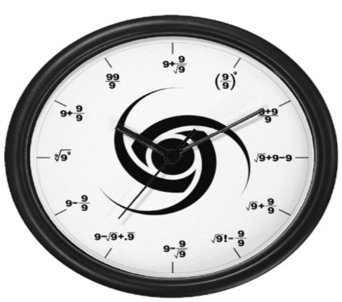
- Formula predicts at most 7680 at level 1
- Since digits are repeated, at most 320 at level 1
- Thanks to commutativity and coincidence, exactly 25 at level 1
- And at levels 2.0-2.8, in the thousands…

Every Integer is Solvable!

Proof:

$$4-4-\log\sqrt{\left(\frac{4}{4}\right)\%\cdots\%}=-\log\left(10^{-2n}\right)^{1/2}=n$$

How Maniacal are You?



How Maniacal are You?

- Apprentices: work sporadically or occasionally
- Journeymen: have worked consistently for at least six months
- Masters: work years, usually systematically, and may often use the computer

How to Become a Master

- Know singles and pairs
- Record everything you find
- Don't just try for the next number
- Work all variations on a pattern
- Understand how operations work
- Use the computer

How Operations Work

- Binary ops: 3 + 5, $\sqrt[3]{8}$, ${}_{4+7}C_3$
- Unary ops: 6%, 4!, $\sqrt{4}$, sinh(ln 2)
- Limited ops:

$$2.34 = \quad 2,34, \quad = 2\frac{34}{100} = \frac{234}{100}$$

$$2.\overline{34} = \quad 2,,34 \quad = 2\frac{34}{99} = \frac{232}{99}$$

$$2.3\overline{4} = \quad 2,3,4 \quad = 2\frac{3}{10} + \frac{4}{90} = \frac{211}{90}$$

Hash Tables

- Create a table for each possible subset of digits: {4}, {4, 4}, {4, 4, 4}, {4, 4, 4, 4}
- Style: {integer, solution string, level}
- Include limited ops in each hash table
- Do unary ops on each hash table
- Do binary ops on the "smaller" hash tables to produce entries for the "larger" hash tables

Hash Table Examples

- { 4, 4, 1.0 }
- { .4, .4, 2.2 }
- { 4/9, .4~, 2.4 }
- { 2, sqrt(4), 3.2 }
- { 2/3, sqrt(.4~), 3.6 }
- { 24, 4!, 3.2 }
- { 6, Γ(4), 5.2 }
- { 1, Γ(sqrt(4)), 5.4 }
- { 120, Γ(Γ(4)), 5.4 }
- { 720, Γ(4)!, 5.4 }
- { 0, log(Γ(sqrt(4))), 5.6 }
- { 44, 44, 2.0 }
- { 4.4, 4.4, 2.0 }
- { 0, 4 – 4, 1.0 }
- { 1, 4 / 4, 1.0 }
- { 8, 4 + 4, 1.0 }
- { 16, 4 x 4, 1.0 }
- { 10, 4 / .4, 2.2 }
- { 100, 4 /(4%), 2.2 }
- { 9, 4/.4~, 2.4 }

Care to Contribute?

4294967320 (3.2)
$4^{4\times 4} + 4!$
Josh Crow, 2/09
Lenexa, KS

38862602496 (3.0)
444^4
Jim Bowdler, 10/07
Solihull, England

googol (3.4)
$\left(\frac{4}{.4}\right)^{4/(4\%)}$
Brian Hollenbeck, 5/03
Emporia, KS

googolplex (5.8)
$\left(\frac{4}{.4}\right)^{\sqrt[-4\%]{\ln \operatorname{arcsec} \sqrt{\sqrt{4}}\,\%\%}}$
Ralph Jeffords, 2/09
Centreville, VA

Conference Special

- Get recognition for your work today!
- Submit your name, hometown, and some of your solutions before you leave, and they may be posted on the Integermania website.
- http://www.milefoot.com/math/integermania
- Fine print: Only one solution per integer will be posted. At most five solutions per person will be posted. The most exquisite solutions will be preferred. Only one submitter per solution will be posted. For this special offer, the webmaster reserves the right to decide which solutions will be posted.