Counting Parenthesizings with Decimals

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Kansas-MAA, May 2009
Pittsburg, Kansas

The Order of Operations Is Important

- 5 – 3 x 8 + 4 / 2 = –17
- (5 – 3) x 8 + 4 / 2 = 18
- 5 – 3 x (8 + 4) / 2 = –13
- (5 – 3 x 8 + 4) / 2 = –7.5
- 5 – (3 x 8 + 4) / 2 = –9
- (5 – 3) x (8 + 4) / 2 = 12
- 5 – 3 x (8 + 4 / 2) = –25
- (5 – 3 x (8 + 4)) / 2 = –15.5

The General Question

- Given a set of n distinct integers, all of which must be used exactly once each (in any order)
- and a set of k (unlimited) binary operations, (including place value, decimals, repeaters), of which $n – 1$ are selected with replacement
- how many solutions could result?

How Many Solutions?

- For the set { 2, 3, 4, 5, 8 }, with operations { +, −, x, ÷ }, there are at most 430,080 solutions.
- In general, with n distinct values and k binary operations, there are at most $\frac{k^{n-1}(2n-2)!}{(n-1)!}$ solutions.

Proof of Formula $\frac{k^{n-1}(2n-2)!}{(n-1)!}$

With n values and k operations, we need the product of the number of ways to ...

- arrange the n values

  $n!$
- choose and arrange the $(n – 1)$ operations

  k^{n-1}
- insert parentheses

  Catalan sequence: $C(n-1)=\frac{1}{n}\binom{2n-2}{n-1}$

Proof, Part 2

- Let P_{n-1} be the number of parenthesizings of $m_1p_1m_2 \ldots m_{n-1}p_{n-1}m_n$
- $(m_1p_1m_2 \ldots p_{q-1}m_q)p_q(m_{q+1}p_{q+1}m_{q+2} \ldots p_{n-1}m_n)$
- Recursive relation:

$$P_{n-1}=\sum_{q=1}^{n-1} P_{q-1}P_{n-q-1}\ ,\ P_0=1$$

$$P_{n+1}=\sum_{q=0}^{n} P_{q}P_{n-q}\ ,\ P_0=1$$

Proof, Part 3

- Consider the power series

$$S(x)=\sum_{n=0}^{\infty}P_n x^n=1+\sum_{n=0}^{\infty}P_{n+1}x^{n+1}$$
$$=1+\sum_{n=0}^{\infty}\left(\sum_{q=0}^{n}P_q P_{n-q}\right)x^{n+1}$$
$$=1+\sum_{q=0}^{\infty}\left(P_q x^{q+1}\sum_{n=q}^{\infty}P_{n-q}x^{n-q}\right)$$
$$=1+x\,S(x)\sum_{q=0}^{\infty}P_q x^q=1+x\left(S(x)\right)^2$$

Proof, Part 4

- The solution of $S(x)=1+x\left(S(x)\right)^2$
- is $S(x)=\frac{1-\sqrt{1-4x}}{2x}$
- whose power series expansion is

$$S(x)=\frac{1}{2x}\sum_{k=1}^{\infty}\frac{2(2k-2)!}{k!(k-1)!}x^k=\sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}x^k$$

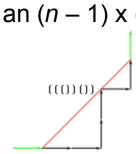
- Thus $P_n=\frac{1}{n+1}\binom{2n}{n}$ or $P_{n-1}=\frac{1}{n}\binom{2n-2}{n-1}$

Alternative Proof (for Parenthesizings)

- Given *n* values with *n* – 1 operations
  example: (((5 – 3) x 8) + (4 / 2))
- there are *n* – 1 left parentheses and *n* – 1 right parentheses (2*n* – 2 total)
- Total number of ways to arrange parentheses (valid or invalid) is $\binom{2n-2}{n-1}$

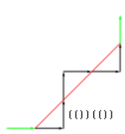
Alternative Proof, Part 2

- If each left parenthesis is a move right,
- and each right parenthesis is a move up
- then valid parenthesizings remain below the diagonal (except for first and last parentheses) in an (*n* – 1) x (*n* – 1) rectangle

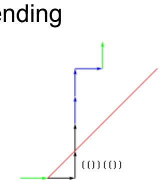


Alternative Proof, Part 3

- and invalid parenthesizings will be above the diagonal in an (*n* – 1) x (*n* – 1) rectangle



- but if the path after the first offending move is flipped, give an (*n* – 2) x (*n* – 0) rectangle



- So the number of invalid parenthesizings is $\binom{2n-2}{n-2}$

Alternative Proof, Part 4

- So the number of valid parenthesizings is

$$\binom{2n-2}{n-1}-\binom{2n-2}{n-2}=\frac{(2n-2)!}{(n-1)!\,(n-1)!}-\frac{(2n-2)!}{(n-2)!\,(n-0)!}$$
$$=\frac{(2n-2)!}{(n-0)!\,(n-1)!}(n-0-n+1)=\frac{1}{n}\binom{2n-2}{n-1}=C(n-1)$$

Actual Solutions?

For the set { 2, 3, 4, 5, 8 },
with operations { +, −, x, ÷ }

- Formula predicts at most 430,080 solutions
- Thanks to field axioms and just plain coincidence, only 4,802 distinct rational solutions (and 1 undefined "solution")
- Of those, 488 are integers

Formula is Least Upper Bound

- Let a, b ϵ { 2, 3, 4, 5, 8}
- Let k ϵ {1, 6, 7, 9}
- Define

$$a \oplus_k b = k \times 10^{2+[\log_{10} a]+[\log_{10} b]} + a \times 10^{1+[\log_{10} b]} + b$$

- Then every solution is unique

Example: $(2 \oplus_6 3) \oplus_7 (5 \oplus_1 (4 \oplus_9 8))$

$$a \oplus_k b = k \times 10^{2+[\log_{10} a]+[\log_{10} b]} + a \times 10^{1+[\log_{10} b]} + b$$

$$2 \oplus_6 3 = 6 \times 10^{2+[\log_{10} 2]+[\log_{10} 3]} + 2 \times 10^{1+[\log_{10} 3]} + 3 = 623$$

$$4 \oplus_9 8 = 9 \times 10^{2+[\log_{10} 4]+[\log_{10} 8]} + 4 \times 10^{1+[\log_{10} 8]} + 8 = 948$$

$$5 \oplus_1 948 = 1 \times 10^{2+[\log_{10} 5]+[\log_{10} 948]} + 5 \times 10^{1+[\log_{10} 948]} + 948 = 15948$$

$$623 \oplus_7 15948 = 7 \times 10^{2+[\log_{10} 623]+[\log_{10} 15948]} + 623 \times 10^{1+[\log_{10} 15948]} + 15948 = 762315948$$

762315948

$$7(6(2,3),1(5,(9(4,8))))$$

$$(2 \oplus_6 3) \oplus_7 (5 \oplus_1 (4 \oplus_9 8))$$

Place Value Concatenation

- 4 *c* 5 = 45
- 2 *c* 4 *c* 5 = 245

- (4 *c* 5) + 8 = 53 is valid
- 4 *c* (5 + 8) is invalid

- Place Value Concatenation (PVC) must precede other operations, it is a limited binary operation

Counting with PVCs

Example: 5 – 3 x 8 *c* 4 / 2 = –121

- One extra factor: locating the PVCs (the PVCs are not counted in variable k)
- 0 PVCs: $n!\ 1\ k^{n-1}\ C(n-1)$
- 1 PVC: $n!\ (n-1)\ k^{n-2}\ C(n-2)$
- 2 PVCs: $n! \binom{n-1}{2} k^{n-3}\ C(n-3)$
- q PVCs: $n! \binom{n-1}{q} k^{n-q}\ C(n-q-1)$

Totals with PVCs

- Total: $$\sum_{q=0}^{n-1} n!\, k^{n-q} \binom{n-1}{q} C(n-q-1) = \sum_{q=0}^{n-1} \frac{n!\, k^{n-q}}{n-q} \binom{n-1}{q} \binom{2n-2q-2}{n-q-1}$$

- So for the set { 2, 3, 4, 5, 8 }, with operations { +, −, x, ÷ } plus PVC, there are at most 2,435,040 solutions.

What About Decimals and Repeaters?

- Three styles:

$$28 = 28$$
$$2.8 = 2\frac{8}{10} = \frac{28}{10}$$
$$2.\overline{8} = 2\frac{8}{9} = \frac{26}{9}$$

- And three more:

$$.28 = \frac{28}{100}$$
$$.\overline{28} = \frac{28}{99}$$
$$.2\overline{8} = \frac{2}{10} + \frac{8}{90} = \frac{26}{90}$$

More Decimals and Repeaters

- With one digit

$$2 = 2$$
$$.2 = \frac{2}{10}$$
$$.\overline{2} = \frac{2}{9}$$

- With three digits

$$234 \quad 2.\overline{34}$$
$$23.4 \quad .234$$
$$23.\overline{4} \quad .23\overline{4}$$
$$2.34 \quad .2\overline{34}$$
$$2.3\overline{4} \quad .\overline{234}$$

Counting All Limited Operations

- Each can be thought of as having 2 points, one decimal point and one repeater point

$$234 = 234.. \quad 2.3\overline{4} = 2.3.4$$
$$23.4 = 23.4. \quad .\overline{234} = ..234$$

- Analogous to 2 indistinguishable balls into d + 1 distinguishable urns, so the number of variations for a d-digit number is

$$\binom{b+u-1}{b} = \binom{d+2}{2}$$

Partitioning Needed

- For n values, we now need to know the partitions of n.
- For example, for the set { 2, 3, 4, 5, 8 }, n = 5
- Seven partitions of 5:

5	4+1	3+1+1
3+2	2+1+1+1	
2+2+1	1+1+1+1+1	

Limited Operation Counting

- Identify **partitions**
- **Assign** values to partition elements
- **Arrange** values in each partition element
- Add decimal and repeater **points**
- Count solutions with earlier **formula**
- **Total**

The Work

Partition	Assign	Arrange	Points	Formula	Total
5	1	$5!$	$\binom{7}{2}$	$1!\,4^{1-1}C(1-1)$	2520
$4+1$	$\binom{5}{4}$	$4!$	$\binom{6}{2}\binom{3}{2}$	$2!\,4^{2-1}C(2-1)$	43200
$3+1+1$	$\binom{5}{3}$	$3!$	$\binom{5}{2}\binom{3}{2}^2$	$3!\,4^{3-1}C(3-1)$	1036800
$3+2$	$\binom{5}{3}$	$3!2!$	$\binom{5}{2}\binom{4}{2}$	$2!\,4^{2-1}C(2-1)$	57600
$2+1+1+1$	$\binom{5}{2}$	$2!$	$\binom{4}{2}\binom{3}{2}^3$	$4!\,4^{4-1}C(4-1)$	24883200
$2+2+1$	$\binom{5}{2}\binom{3}{2}$	$2!2!$	$\binom{4}{2}^2\binom{3}{2}$	$3!\,4^{3-1}C(3-1)$	2488320
$1+1+1+1+1$	1	1	$\binom{3}{2}^5$	$5!\,4^{5-1}C(5-1)$	104509440
					133021080

The Answers

- For the set { 2, 3, 4, 5, 8 }, with operations { +, −, x, ÷, *c*, . , ¯ }, the number of solutions is at most

- without limited operations: 430,080
- with PVCs only: 2,435,040
- with all limited operations: 133,021,080

Questions ?

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