Exploring Quartic Plane Curves with *Excel*

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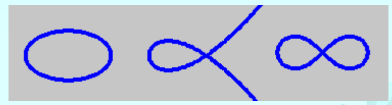
Quartic Plane Curves Outline

- The Conic Motivation
- Mathematical Development
- Technological Implementation
- Exploring Quartics (and Cubics)
- Curricular Situations



Mathematical Development

- Quartic Plane Curve
- In terms of y: Quartic in 1 variable
- Ferrari: 2 quadratics and 1 cubic
- Reduce the Cubic (omit square term)
- Cardano: A cubic solution
- Four Solutions via Quadratic Formula
- Special Cases

Quartic Plane Curve

$$\begin{aligned} &a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4 \\ &+ a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ &+ a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ &+ a_{10}x + a_{01}y \\ &+ a_{00} = 0 \end{aligned}$$

15 coefficients in the general equation

Quartic in One Variable

$$\begin{aligned} &(a_{04})\,y^4 \\ &+ (a_{13}x + a_{03})\,y^3 \\ &+ (a_{22}x^2 + a_{12}x + a_{02})\,y^2 \\ &+ (a_{31}x^3 + a_{21}x^2 + a_{11}x + a_{01})\,y \\ &+ (a_{40}x^4 + a_{30}x^3 + a_{20}x^2 + a_{10}x + a_{00}) = 0 \end{aligned}$$

Given an x-value, solve:

$$a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 = 0$$

Ferrari's Method

$$a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0 = 0$$

$$4\left(y^4 + \frac{a_3}{a_4}y^3 + \frac{a_2}{a_4}y^2 + \frac{a_1}{a_4}y + \frac{a_0}{a_4}\right) = 0$$

Add $(h_1y + h_2)^2$ to both sides, then find h_1, h_2, and k so that:

$$\left(2y^2 + \frac{a_3}{a_4}y + k\right)^2 = (h_1y + h_2)^2$$

Two Quadratics

$$2y^2 + \frac{a_3}{a_4} y + k = \pm\left(h_1 y + h_2\right)$$

$$2y^2 + \left(\frac{a_3}{a_4} \mp h_1\right) y + \left(k \mp h_2\right) = 0$$

or alternatively

$$\left(2y^2 + \left(\frac{a_3}{a_4} + h_1\right) y + \left(k + h_2\right)\right)\left(2y^2 + \left(\frac{a_3}{a_4} - h_1\right) y + \left(k - h_2\right)\right) = 0$$

Equating Coefficients

$$4y^4 + \frac{4a_3}{a_4} y^3 + \left(4k + \frac{a_3^{\ 2}}{a_4^{\ 2}} - h_1^{\ 2}\right) y^2 + \left(\frac{2a_3 k}{a_4} - 2h_1 h_2\right) y + \left(k^2 - h_2^{\ 2}\right) = 0$$

$$a_4 y^4 + a_3 y^3 + \left(a_4 k + \frac{a_3^{\ 2}}{4a_4} - \frac{a_4 h_1^{\ 2}}{4}\right) y^2 + \left(\frac{a_3 k - a_4 h_1 h_2}{2}\right) y + \frac{a_4}{4}\left(k^2 - h_2^{\ 2}\right) = 0$$

Compare with the original:

$$a_4 y^4 + a_3 y^3 + a_2 y^2 + a_1 y + a_0 = 0$$

and equate coefficients

Cubic Condition

$$\begin{cases} a_2 = a_4 k + \dfrac{a_3^{\ 2}}{4a_4} - \dfrac{a_4 h_1^{\ 2}}{4} \\ a_1 = \dfrac{a_3 k - a_4 h_1 h_2}{2} \\ a_0 = \dfrac{a_4}{4}\left(k^2 - h_2^{\ 2}\right) \end{cases} \qquad \begin{cases} h_1^{\ 2} = 4k + \dfrac{a_3^{\ 2}}{a_4^{\ 2}} - \dfrac{4a_2}{a_4} \\ h_1 h_2 = \dfrac{a_3 k - 2a_1}{a_4} \\ h_2^{\ 2} = k^2 - \dfrac{4a_0}{a_4} \end{cases}$$

We can eliminate the variables h_1 and h_2

$$\frac{\left(a_3 k - 2a_1\right)^2}{a_4^{\ 2}} = \left(h_1 h_2\right)^2 = \left(4k + \frac{a_3^{\ 2}}{a_4^{\ 2}} - \frac{4a_2}{a_4}\right)\left(k^2 - \frac{4a_0}{a_4}\right)$$

$$a_4^{\ 3} k^3 - a_2 a_4^{\ 2} k^2 + \left(a_1 a_3 a_4 - 4a_0 a_4^{\ 2}\right) k + \left(4a_0 a_2 a_4 - a_1^{\ 2} a_4 - a_0 a_3^{\ 2}\right) = 0$$

Eliminating the Square Term

$$b_3 k^3 + b_2 k^2 + b_1 k + b_0 = 0$$

Substitute: $k = z - h_3$

$$b_3\left(z - h_3\right)^3 + b_2\left(z - h_3\right)^2 + b_1\left(z - h_3\right) + b_0 = 0$$

$$\begin{cases} b_3 z^3 + \left(b_2 - 3b_3 h_3\right) z^2 + \left(b_1 - 2b_2 h_3 + 3b_3 h_3^{\ 2}\right) z \\ \qquad + \left(b_0 - b_1 h_3 + b_2 h_3^{\ 2} - b_3 h_3^{\ 3}\right) \quad = 0 \end{cases}$$

Choose h_3 so that square term disappears: $h_3 = \dfrac{b_2}{3b_3}$

The Reduced Cubic

$$b_3 z^3 + \left(b_1 - \frac{b_2^{\ 2}}{3b_3}\right) z + \left(b_0 - \frac{b_1 b_2}{3b_3} + \frac{2b_2^{\ 3}}{27b_3^{\ 2}}\right) = 0$$

$$z^3 + \frac{3b_1 b_3 - b_2^{\ 2}}{3b_3^{\ 2}} z + \frac{27b_0 b_3^{\ 2} - 9b_1 b_2 b_3 + 2b_2^{\ 3}}{27b_3^{\ 3}} = 0$$

$$z^3 + h_4 z + h_5 = 0$$

Cardano's Method

$$z^3 + h_4 z + h_5 = 0$$

Substitute: $z = u + v$

$$\left(u + v\right)^3 + h_4\left(u + v\right) + h_5 = 0$$

$$u^3 + v^3 + \left(3uv + h_4\right)\left(u + v\right) + h_5 = 0$$

Choose v so the linear terms disappear: $v = \dfrac{-h_4}{3u}$

$$u^3 + \left(\frac{-h_4}{3u}\right)^3 + h_5 = 0$$

$$27u^6 + 27h_5 u^3 - h_4^{\ 3} = 0$$

Solving for *u*

By the quadratic formula:

$$u^3 = \frac{-h_5}{2} \pm \sqrt{\left(\frac{-h_5}{2}\right)^2 + \left(\frac{h_4}{3}\right)^3}$$

$$\text{Define} \begin{cases} g_1 = \dfrac{h_4}{3} = \dfrac{3b_1b_3 - b_2^2}{9b_3^2} \\ g_2 = \dfrac{-h_5}{2} = \dfrac{9b_1b_2b_3 - 27b_0b_3^2 - 2b_2^3}{54b_3^3} \\ \Delta_3 = -g_1^3 - g_2^2 \quad \text{(the discriminant)} \end{cases}$$

$$u = \sqrt[3]{g_2 \pm \sqrt{-\Delta_3}}$$

Solving for *v*

Since $u^3 = g_2 \pm \sqrt{-\Delta_3}$, then:

$$v^3 = \left(\frac{-h_4}{3u}\right)^3 = \frac{-g_1^3}{g_2 \pm \sqrt{-\Delta_3}}\left(\frac{g_2 \mp \sqrt{-\Delta_3}}{g_2 \mp \sqrt{-\Delta_3}}\right)$$

$$v^3 = \frac{-g_1^3}{g_2^2 + \Delta_3}\left(g_2 \mp \sqrt{-\Delta_3}\right) = \frac{-g_1^3}{-g_1^3}\left(g_2 \mp \sqrt{-\Delta_3}\right)$$

$$v^3 = g_2 \mp \sqrt{-\Delta_3}$$

u^3 and v^3 are conjugates

$$v = \sqrt[3]{g_2 \mp \sqrt{-\Delta_3}}$$

Cubic Solution: $\Delta_3 \le 0$

$$\text{Define} \begin{cases} g_3 = \sqrt[3]{g_2 + \sqrt{-\Delta_3}} \\ g_4 = \sqrt[3]{g_2 - \sqrt{-\Delta_3}} \\ g_5 = -h_3 = \dfrac{-b_2}{3b_3} \end{cases}$$

$$k = z - h_3 = u + v + g_5$$

only one real solution, only one is needed

$$k = g_3 + g_4 + g_5$$

Solving Trigonometrically: $\Delta_3 > 0$

$$u^3 = g_2 + \sqrt{-\Delta_3} = g_2 + i\sqrt{\Delta_3}$$

$$\left|u^3\right| = \sqrt{g_2^2 + \left(\sqrt{\Delta_3}\right)^2} = \sqrt{-g_1^3} = -g_1\sqrt{-g_1}$$

$$\theta = \arg u^3 = \arccos \frac{g_2}{-g_1\sqrt{-g_1}}$$

$$u^3 = -g_1\sqrt{-g_1}\left(\cos\theta + i\sin\theta\right)$$

$$u = \sqrt{-g_1}\left(\cos\frac{\theta + 2n\pi}{3} + i\sin\frac{\theta + 2n\pi}{3}\right)$$

Cubic Solutions: $\Delta_3 > 0$

u and *v* are conjugates

$$u + v = 2\sqrt{-g_1}\left(\cos\frac{\theta + 2n\pi}{3}\right)$$

$$k = z - h_3 = u + v + g_5$$

$$k = 2\sqrt{-g_1}\left(\cos\frac{\theta + 2n\pi}{3}\right) + g_5, \text{ where } \theta = \arccos\frac{g_2}{-g_1\sqrt{-g_1}}$$

three real solutions, the quartic uses the largest

$$k = \max_{n=0,1,2}\left\{2\sqrt{-g_1}\left(\cos\frac{\theta + 2n\pi}{3}\right) + g_5\right\}$$

Values dependent on *k*

$$\text{Let} \begin{cases} g_6 = h_1^2 = 4k + \dfrac{a_3^2}{a_4^2} - \dfrac{4a_2}{a_4} \\ g_7 = h_2^2 = k^2 - \dfrac{4a_0}{a_4} \\ g_8 = -h_1h_2 = \dfrac{2a_1 - a_3k}{a_4} \end{cases}$$

$$\text{Then } h_1 = \sqrt{g_6} \text{ and } h_2 = \begin{cases} \sqrt{g_7} & \text{when } g_8 < 0 \\ -\sqrt{g_7} & \text{when } g_8 \ge 0 \end{cases}$$

Quadratic Coefficients

$$2y^2 + \left(\frac{a_3}{a_4} \mp h_1\right) y + \left(k \mp h_2\right) = 0$$

$$2y^2 + \left(\frac{a_3}{a_4} \mp \sqrt{g_6}\right) y + \left(k \mp \left(\pm\sqrt{g_7}\right)\right) = 0$$

Use quadratic formula with:

$$\begin{cases} c_{12} = 2 \\ c_{11} = \dfrac{a_3}{a_4} - \sqrt{g_6} \\ c_{10} = \begin{cases} k - \sqrt{g_7} & \text{if } g_8 < 0 \\ k + \sqrt{g_7} & \text{if } g_8 \ge 0 \end{cases} \end{cases} \quad \text{and} \quad \begin{cases} c_{22} = 2 \\ c_{21} = \dfrac{a_3}{a_4} + \sqrt{g_6} \\ c_{20} = \begin{cases} k + \sqrt{g_7} & \text{if } g_8 < 0 \\ k - \sqrt{g_7} & \text{if } g_8 \ge 0 \end{cases} \end{cases}$$

$$\Delta_{2i} = c_{i1}^{\,2} - 4c_{i2}c_{i0}$$

The Four Solutions

$$y_1 = \frac{-c_{11} + \sqrt{\Delta_{21}}}{2c_{12}} \quad \text{if } \Delta_{21} \ge 0$$

$$y_2 = \frac{-c_{11} - \sqrt{\Delta_{21}}}{2c_{12}} \quad \text{if } \Delta_{21} \ge 0$$

$$y_3 = \frac{-c_{21} + \sqrt{\Delta_{22}}}{2c_{22}} \quad \text{if } \Delta_{22} \ge 0$$

$$y_4 = \frac{-c_{21} - \sqrt{\Delta_{22}}}{2c_{22}} \quad \text{if } \Delta_{22} \ge 0$$

Special Cases

- If $a_4=0$, then let $b_i=a_i$, go to cubics
- If $a_4=0$, $b_3\neq0$, then display cubic solutions
- If $a_4=b_3=0$, then let $c_i=b_i$, go to quadratics
- If $a_4=b_3=c_{i2}=0$, then solve linear equation: $y_5 = -a_0/a_1$

Technological Implementation

- Spreadsheet: MS *Excel*
- 15 sliders for 15 coefficients
- Real-time parameter variation (ActiveX, not Forms)
- Graphs on [-10,10]x[-10,10]
- Properly drawn graphs

Technological Issues

- Domain
  - Zero or negative parameters
  - Sign condition
- Continuity
  - Dot vs. connected mode
- Numerical Precision
  - Parameter accuracy
  - Number of solutions

Domain Issues

- MS *Excel* error situations:
  - #DIV/0! (division by zero)
  - #NUM! (non-real number)
- MS *Excel* graph behavior:
  - Plot as if zero
  - Produce false branches

Division by Zero

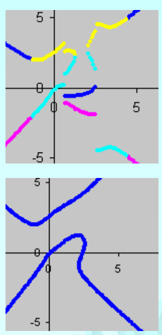
- Variables: a_4, a_3, a_2, a_1, b_3, c_{i2}
- Reason 1: Degree classification
  - Branch in logical reasoning
- Reason 2: Undefined at one value
  - Redefine solution offscreen (y_i=22)

Negative Discriminants

- Variables: Δ_3, Δ_{2i}
- Reason 1: Character of solution method
  - Branch in logical reasoning
- Reason 2: Character of solutions
  - Redefine solution offscreen (y_i=22)

Sign Condition

- Variable: g_8
- Pairing of quadratic coefficients



Continuity Issues

<u>Dot Mode?</u>

\+ No false pieces

– Gaps

? <u>Minimize dot gaps</u>

? Steepest graph

? x-increment

<u>Continuous Mode?</u>

\+ No gaps

– False pieces

? <u>Avoid false pieces</u>

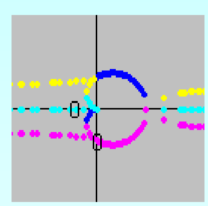
? Domain formulas

Dot Mode x 2!

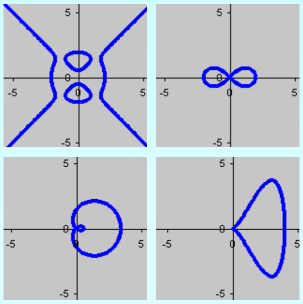
- y in terms of x AND x in terms of y
- Largest gap when slope is 1
- Avoids false pieces
- Bonus: Vertical lines graphed

Numerical Imprecision

- *Excel*: 15 digits
- Δ_3 can exceed 10^{23}
- Subtraction!
- Observed errors:
  - Δ_3, arg(θ), g_6, g_7, Δ_{21}, Δ_{22}
- Solution: rounding

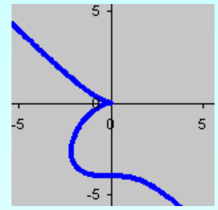


Let's Explore!



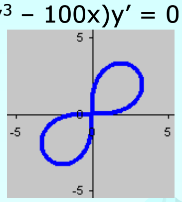
Exploration: Beyond Conics

- Describe the changes in the graph of $x^3 + y^3 + ay^2 = 0$ when the parameter is varied.
- Classify the types of graphs available from the general cubic equation.



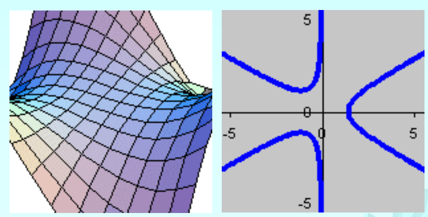
Interpretation: Calculus & ODEs

- Find the slope of the graph of $3(x^2 + y^2)^2 = 100xy$ at (3,1).
- Given y(3)=1, solve the exact ODE $(12x^3 + 12xy^2 - 100y)+$ $(12x^2y + 12y^3 - 100x)y' = 0$.



Descriptions: 3D Cross-Sections

- Monkey Saddle
  - $z = x^3 - 3\,x\,y^2$



Web Site

- http://staff.jccc.edu/swilson/planecurves/quarticplane.xls
- Macro Security Level ≤ Medium